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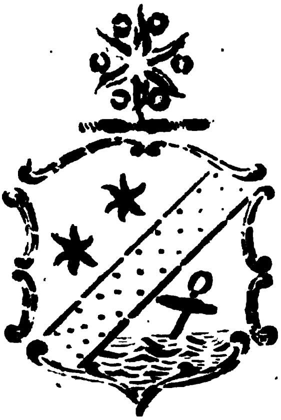

George Croxton Shiffner.

A
COURSE
OF
MATHEMATICS,

DESIGNED FOR THE USE
OF THE
OFFICERS AND CADETS,
OF THE
ROYAL MILITARY COLLEGE.

By ISAAC DALBY,
Professor of Mathematics in the said College.

VOL. II.

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PREFACE TO VOL. II.

The subjects composing this Volume are *Algebra*, *Conic Sections*, *Mechanics*, *Hydrostatics*, and *Pneumatics*. But the particulars are enumerated in the table of contents which renders a detail in this place unnecessary. We therefore have little more than a few detached observations to offer by way of preface.

After Fractions, the arrangement in Algebra is not exactly similar to that usually found in more extensive treatises. Some reasons are given for a particular deviation (*p.* 82). And it is from considerable experience in teaching that we were induced to prefer the order in which the several rules or parts follow one another. Learners however, generally consider the management of radical quantities as the most difficult task in Algebra: a master therefore may sometimes perceive the necessity of bringing a student forward to a particular extent in Quadratic Equations before he enters upon Surds.

The examples of extracting roots, Art. 109, may not be thought sufficient to illustrate the general rule; but the algebraist will remember that the two right hand figures of the dividends in extracting the cube root, three in extracting the 4th root, and the four right hand figures in those of the 5th root, &c. are always neglected in making the division. But the method derived from Dr. Halley's formulæ is preferable to all others.

After the Algebra was drawn up, we found an investigation of the Binomial Theorem in M. Dubourguet's Algebra by means of the same artifice as that in the work quoted *p.* 149; the author has extended the demonstration to fractional and negative indices.

The different series to which Art. 172—181 are an introduction, may be reckoned among the speculative parts of mathematics. The principal theorem in the Arithmetic of Infinites however, is deduced from the Differential Method, (Art. 179); the application of this formula has been of considerable use in the subsequent part of the volume.

In treating of the Conic Sections, the fundamental property or the equation of each curve is derived from the solid: afterwards they are considered in plano; and as the expressions for the ellipse and hyperbola differ in nothing but the signs + and —, the same demonstration frequently answers for both sections by only changing those signs; for which reason the enunciations of some properties of the hyperbola are thought sufficient.

That part of Mechanics which relates to the Centre of Gravity is given at some length on account of its extensive use. In Art. 386, 387, 388, different methods of computing the thickness of walls or revetments are compared. The results, as might be expected from different hypotheses, vary considerably. By adopting the method however, in the first work refered to (*p.* 383), we evidently are led to the following conclusion (p. 386) which is correct only in the case of a fluid,—namely, that the lateral pressure of a body of loose earth depends on its height without any regard to thickness.—But as all the computations are founded upon uncertain data, no correction of principle is attempted: and the only alteration is that of giving a more convenient form to M. Belidor's solution, which, as it nearly agrees with the practice of Vauban, seems the least liable to exception. All theories however, respecting the strength of walls, and also that of *timber*, must necessarily be imperfect. On the latter subject, see an account of the very extensive and laborious experiments of M. de Buffon in *Mem. Acad. des Sciences*, 1740.

The speculative mechanician therefore will seldom find an exact agreement between his conclusions and the results from experiment; particularly in what relates to the working of machinery, because no theory of Friction has yet been discovered by which its effects can be calculated; for that reason the subject is not considered in the following pages.

As a work of this kind must unavoidably consist of abridgements, considerable care was bestowed in selecting what appeared the most useful to students who have not an opportunity of perusing the separate and more diffuse treatises on the different subjects. Some new solutions are introduced: but the mathematical reader cannot expect much new matter in any form.

To conclude.—The experience of two or three years proves that it will not be necessary to extend the Course beyond this Volume for the use of the College. Those Officers or Cadets who may gain a thorough knowledge of the principal matters contained in both volumes during their stay, and are inclined to continue the study of mathematics after quitting the Institution, will consult books professedly written on the higher branches, and pursue their researches without the assistance of a master.

High Wycomb,
June 6, 1805.

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ALGEBRA.

DEFINITIONS AND NOTATION.

1. ALGEBRA is a science which treats of the properties of numbers in general, by means of the numeral figures and other symbols; it is therefore called *Universal Arithmetic:* and sometimes *Analysis*, or, the *Analytic Art.*

2. In the operations by common Arithmetic we employ the numeral figures only, but in Algebra it is usual to represent quantities of every kind, both known and unknown, by letters of the Alphabet; and in this consists its peculiar excellence, because the reasoning is carried on with the letters or symbols, whose values are to be found, in the same manner as with those which denote given quantities.

3. The initial letters of the alphabet, a, b, c, d, &c. are commonly put for given quantities; and the final letters, x, y, z, v, &c. for those that are unknown, or required.

4. The leading marks or characters of abbreviation are given in the Arithmetic; but a more particular explanation will be necessary.

$+$ *plus,* signifies *addition:* this is called the *affirmative* or *positive* sign.

$-$ *minus,* signifies *subtraction*; the *negative* sign.

$\times$ *(into),* signifies *multiplication.*

$\div$ *division.*

$=$ *equal to,* the mark for *equality.*

Thus $a + b$ is read a plus b, and denotes the sum of a and b.
Let $a = 8$, $b = 3$, and $c = 11$:
then $a + b = c$ (a plus b equal to 11) or $8 + 3 = 11$.

$a - b$ is read a minus b, and shews that the quantity represented by b is to be subtracted from that denoted by a.
If $a = 8$, $b = 3$, and $d = 5$:
then $a - b = d$ (a minus b is equal to d) or $8 - 3 = 5$.

$a \times b$ (a into b) denotes the product of a and b.
If $a = 8$, $b = 3$, and $c = 24$:
then $a \times b = c$, or $8 \times 3 = 24$. Here a and b are the factors of the product ab.

But simple factors, as a, b, c, &c. are usually placed without any mark or sign between them, to denote their product: sometimes however, a full stop is used:

Thus $a \times b$, or ab, or $a.b$, signify the product of b and a.
And $4abc$ the continued product of the factors 4, a, b, c.
If $a = 2$, $b = 3$, and $c = 5$; then $4abc = 4 \times 2 \times 3 \times 5 = 120$.

$a \div b$ shews that a is to be divided by b. But the usual form of setting down the quotient is that of a vulgar fraction; thus $\frac{a}{b}$, which denotes the quotient of a divided by b (37. Arith.)

Also $\frac{a+b}{a-c}$ denotes the quotient of $a+b$ divided by $a-c$.

Let $a=4$, $b=5$, and $c=2$; then $a \div b$, or $\frac{a}{b}=\frac{4}{5}$.

And $\frac{a+b}{a-c}=\frac{4+5}{4-2}=\frac{9}{2}$.

5. An Equation is known by the symbol $=$ (equal to):
Thus $x=a+b$ is an equation, shewing that x is equal to the sum of a and b.

6. The character $\backsim$ denotes the *difference* of two quantities when it is not known which is the greatest:
Thus, $a \backsim b$ denotes the difference of a and b.

7. Proportions are set down as in Arithmetic (92. Arith.)
Thus $a : b :: c : d$, are read, as a is to b, so is c to d.

8. $>$ denotes *greater;* and $<$ *less:*
Thus $a > b$ signifies that a is greater than b:
and $c < a$, that c is less than a.

9. A bar or line drawn over several quantities, denotes that they are to be taken collectively; this is called a *vinculum:* a parenthesis or brackets are also used for the same purpose:

Thus $\overline{a+b} \times \overline{c-d}$
$\overline{a+b} \cdot \overline{c-d}$
$(a+b) \times (c-d)$
$(a+b) \cdot (c-d)$
$(a+b)\ (c-d)$, all denote the same thing; namely, that the sum of a and b is to be multiplied by the difference $c-d$.

And $(a+b)-(-d+c)$ denotes that $-d+c$ is to be subtracted from $a+b$.

10. The *Co-efficient* of any term is the number or known quantity prefixed to it, and denotes how many times it is taken:

Thus 4 is the co-efficient in the quantity $4a$.

And in the quantity $3bx$, if b is given, and x is unknown, then $3b$ is the co-efficient of the unknown quantity x.

Also in $10\,(a+b)$, the co-efficient of $a+b$ is 10.

And 1 is the co-efficient of a, for $1 \times a$ is a.

Also $\frac{1}{4}$ is the co-efficient of $\frac{a}{4}$, for $\frac{1}{4}a$ is the same as $\frac{a}{4}$.

11. *Powers* are denoted by a small figure, called the index or exponent (111. Arith.)

Thus a^2 (the square of a) is the same as $a \times a$, or aa.

a^3 the same as $a \times a \times a$, or aaa.

a^n denotes a raised to the nth. power.

$(b+c-d)^5$ denotes the 5th. power of the compound quantity $b+c-d$: and $(b+c-d)^n$ its nth. power.

12. $\sqrt{\;}$ is the *radical* sign, and shews that the root is to be taken.

Thus $\sqrt{81}$ is 9, viz. the square root of 81 is 9.

$\sqrt[3]{27} = 3$, or the cube root of 27 is 3.

$\sqrt[4]{a^4} = a$, or the 4th root of a^4 is a.

$\sqrt[3]{(a+b)}$ denotes the cube or 3d. root of $a+b$.

$\sqrt[n]{(a^2+b^3)}$, the nth root of a^2+b^3.

But fractional indices are rather more commodious;

Thus $(a+b)^{\frac{1}{2}}$ is the same as $\sqrt{(a+b)}$, each denoting the square root of $a+b$.

$(a+b)^{\frac{1}{3}}$ is the same as $\sqrt[3]{(a+b)}$.

$(a^2+b^3)^{\frac{1}{n}}$, the same as $\sqrt[n]{(a^2+b^3)}$.

13. A *rational quantity* is that which has no radical sign ($\surd$) or index annexed to it, as b, or $5ca$.

14. A *surd* or *irrational quantity* is that which has not an exact root, as $\sqrt{5}$, or $\sqrt{a^3}$, or $(a+b)^{\frac{1}{3}}$.

15. The *reciprocal* of any quantity is 1 divided by that quantity:

Thus, the reciprocal of b is $\frac{1}{b}$.

and the reciprocal of $\frac{a}{c}$ is $\frac{c}{a}$.

16. *Like* or similar quantities are such as differ only in their co-efficients, as $5a$ and a, or $2bc^2x$ and $-3bc^2x$.

17. *Unlike* quantities are those which consist of different letters, or different powers, as $2b$ and a, or $-3a$ and $4a^2$, or $5ab$ and $5ab^2$.

18. *Like signs* are all affirmative ($+$), or all negative ($-$):
Thus, a, b, $a+c$, $4a^2$, are all affirmative, or supposed to have the sign $+$; these are called positive quantities. And $-2ax$, $-3b$, $-a$, have all the negative sign; these are called negative quantities.

19. *Unlike signs* are when some are positive and some negative, as $-4ab+3bc$, or $3x-4b$.

20. *Simple quantities* are those which consist of one term only, as $3ab$, or $\frac{4a}{b}$, or $-7ac^2$.

21. *Compound quantities* consist of several terms, as $a+b$, or $3b-2a$, or $(b+c)\times(d-a)$.

22. A *Binomial* consists of two terms; a *trinomial* of three; a *Quadrinomial* of four, &c.

Thus $a+b$ is binomial:
$a-b+c$, a trinomial:
$2a-2x+c-4d$, a quadrinomial.

23. A *Residual quantity* is a binomial having one of its terms negative, as $2a-c$.

24. *Composite quantities* are those produced by the multiplication of two or more terms called its *factors:*

Thus $3abc$ is a composite quantity, its factors being 3, a, b, c.

And $3abc$ is the common multiple of 3, a, b, c.

25. *Given quantities* are those whose values are known.

26. *Unknown quantities* are those whose values are unknown, or required.

N. B. By *quantities*, in these Definitions, we understand such magnitudes as can be represented by numbers.

27. *Examples* illustrating the method of representing, or combining numbers or quantities algebraically.

Let $a=5$ $\quad c=1$
$b=4$ $\quad d=0.$

Then $4a^2-6abc+ab^2=100-120+80=60.$

$(a^2-b^2)\times(a-b)=\overline{25-16}\times\overline{5-4}=9\times1=9.$

$7(b-c)\ (2a-2bc)=21\times2=42.$

$7(b-c)\times2a-2bc=21\times10-8=210-8=202.$

$\frac{b^3-a^2}{b+d}\times(c-d)+\frac{b}{a}=\frac{64-25}{4+0}\times1+\frac{4}{5}=9\frac{3}{4}+\frac{4}{5}=10\frac{11}{20}.$

$(a-b\times c^2+d^2)^3=1.$

$\frac{a^2b^2}{ab}\times cd+\frac{a}{b}\times\frac{b}{a}=\frac{400}{20}\times0+\frac{5}{4}\times\frac{4}{5}=0+\frac{20}{20}=1.$

$(a^2-b^2)^{\frac{1}{2}}\times(a+b)=(25-16)^{\frac{1}{2}}\times9=3\times9=27.$

$(a^2 - b^2)^{\frac{1}{2}} \times a + b = (25 - 16)^{\frac{1}{2}} \times 5 + 4 = 3 \times 5 + 4 = 19.$

$ab - (a - b - c + d) = 20 - 0 = 20.$

$ab - (b^2 + a - c) = 20 - 20 = 0.$

$\frac{3abc}{\sqrt{(ab + b^2)}} \times (a + b + c) = \frac{60}{6} \times 10 = 100.$

$(\overline{ab^2 + c} \cdot \overline{b - c})^{\frac{1}{3}} = (81 \times 3)^{\frac{1}{3}} = \sqrt[3]{243} = 7.$

$(a + b) \cdot (a - b) - (a + b + c) = 9 \times 1 - 10 = -1.$

$\sqrt{ab} \times \sqrt{ab} = \sqrt{20} \times \sqrt{20} = 20.$

$\sqrt{bc} \times \sqrt{ab} = \sqrt{4} \times \sqrt{20} = 2\sqrt{20}.$

$(3ab + b)^{\frac{2}{3}} = 64^{\frac{2}{3}} = \sqrt[3]{64^2} = 4096^{\frac{1}{3}} = 16.$

$b \backsim a = 1.$

$a \pm b$, or a *plus* and *minus* $b = 5 \pm 4 = 9$, and 1.

$x = ab - c$, or $x = 20 - 1$, or $x = 19.$

$[\sqrt{(a^2 - b^2)} \times (d^2 + 8d)]^{\frac{1}{3}} = (\sqrt{(25 - 16)} \times 9)^{\frac{1}{3}} = (3 \times 9)^{\frac{1}{3}} = 3.$

ADDITION.

28. THE Addition of Algebraic quantities is performed by collecting those that are alike or similar into one sum, and setting down that sum, together with the unlike quantities, all with their proper signs.

29. When the quantities are alike, and have like signs, add their co-efficients together, and prefix the sum to the letter or letters common to each term.

Thus, suppose the sum of the affirmative quantities $2a$, $5a$, $7a$, a, and $3a$ is required:

$$\begin{array}{rr} & 2a \\ & 5a \\ & 7a \\ & a \\ & 3a \\ \hline \text{Sum} & 18a \end{array}$$

Here it is evident, whatever be the value of a, that the sum $2a + 5a + 7a + a + 3a$ is $18a$; where 18 is the sum of the co-efficients.

$$\begin{array}{lr}
\text{Let } a = 4: \text{ then} & 2a = 8 \\
 & 5a = 20 \\
 & 7a = 28 \\
 & a = 4 \\
 & 3a = 12 \\
\hline
 & 72 = 18a = 18 \times 4.
\end{array}$$

If the quantities were negative, or $-2a$, $-5a$, $-7a$, $-a$, $-3a$, the sum would be denoted by $-18a$.

30. When quantities are alike, but have different signs, take the sums of the affirmative and negative co-efficients, respectively, and subtract the less from the greater, then prefix the sign of the greater to the difference, and subjoin the common quantity.

Let the sum of $2ab - ab$, $4ab - 2ab$, $7ab - 5ab$, and $ab - ab$, be required:

$$\begin{array}{rcl}
2ab & - & ab \\
4ab & - & 2ab \\
7ab & - & 5ab \\
ab & - & ab \\
\hline
14ab & - & 9ab = 5ab \text{ the sum.}
\end{array}$$

Now $14 - 9 = 5$, therefore, the sum is $+5ab$ or $5ab$.

$$\begin{array}{lrcl}
\text{Let } a = 2,\ b = 1: & & & \\
\text{Then} & 2ab - ab & = & 4 - 2 \\
 & 4ab - 2ab & = & 8 - 4 \\
 & 7ab - 5ab & = & 14 - 10 \\
 & ab - ab & = & 2 - 2 \\
\hline
 & & & 28 - 18 = 10 = 5ab \text{ the sum.}
\end{array}$$

This process, however, in which subtraction is blended with addition, is evidently nothing more than adding together several differences:

$$\begin{array}{lrcl}
\text{For} & 2ab - ab & = & ab \\
 & 4ab - 2ab & = & 2ab \\
 & 7ab - 5ab & = & 2ab \\
 & ab - ab & = & 0 \\
\hline
 & & & 5ab \text{ the sum of the differences, as before.}
\end{array}$$

31. If the negative coefficients together exceed the affirmative ones, the sum will be negative:

$$\begin{array}{rl}
\text{Thus, the sum of} & -2ab + ab \\
 & -4ab + 2ab \\
 & -7ab + 5ab \\
 & -\ ab + ab \\
\hline
\text{is} & -14ab + 9ab, \text{ or } -5ab.
\end{array}$$

32. When the positive and negative coefficients are equal, the sum becomes $= 0$:

Thus, let the quantities be $2x + a - 3x$, $x - 4x + a$, and $5x - 2a - x$;

$$\begin{array}{rl}
 & 2x - 3x + a \\
 & x - 4x + a \\
 & 5x - x - 2a \\
\hline
\text{sum} & 8x - 8x + 2a - 2a = 0.
\end{array}$$

For $8x - 8x = 0$; and $2a - 2a = 0$.

33. Sometimes the terms may be collected mentally without setting them down one under another.

Thus, suppose the quantities to be added or abridged are $3\sqrt{ax} + 7$, $2\sqrt{ax} - 20$, and $18 - \sqrt{ax}$; then the order of setting down the sum may be

thus, $5\sqrt{ax} - \sqrt{ax} + 7 + 18 - 20$

or, $4\sqrt{ax} + 25 - 20$, which is $4\sqrt{ax} + 5$, the sum.

34. When quantities are unlike, and have like, or different signs, collect those that are similar together, as in the foregoing examples, then set the whole down with their proper signs.

Thus, if the terms are $-ba$ and $+ca$, they may be set down

thus, $-ba + ca$

or thus, $ca - ba$

or thus, $(c - b)\,a$.

Let $a = 4$, $b = 2$, $c = 5$:

then, $ca - ba = 20 - 8 = 12$

or, $(c - b)\,a = (5 - 2) \times 4 = 12$.

It therefore appears, that when the quantities are all unlike, the number of terms cannot be abridged, which is also evident from the following example:

$$\begin{array}{rl} & ax - bx \\ & -cx + dx \\ & gx - hx \\ \hline \text{sum} & ax + dx + gx - bx - cx - hx \\ \text{or,} & (a + d + g - b - c - h)\,x. \end{array}$$

35. If however, a, d, g, b, c, h (the coefficients of x) are given quantities, the whole may be reduced to one term:

Thus, let $a = 6$, $d = 10$, $g = 5$, $b = 12$, $c = 4$, $h = 3$:
then $(a + d + g - b - c - h)\,x$ will be $(21 - 19)\,x$ or $2x$.

Other Examples.

$$\begin{array}{l} -2\,xy + c - 7 \\ -4\,c - xy + 9 \\ axy + \tfrac{1}{2}xy - 3 \\ \hline (a - 2\tfrac{1}{2})\,xy - 3c - 1 \\ \hline \end{array}$$

$$\begin{array}{l} \sqrt{a} - \sqrt{x} + 2a \\ b\sqrt{x} - \tfrac{1}{2}\sqrt{a} - 2ba \\ \hline (b - 1)\sqrt{x} + \tfrac{1}{2}\sqrt{a} + (2 - 2b)\,a. \\ \hline \end{array}$$

$$\begin{array}{l} axy - 14 - \sqrt{z} \\ -2xy + 2z^{\frac{1}{2}} + 15 \\ \tfrac{1}{2}xy - z^2 - 1 \\ \hline (a - 1\tfrac{1}{2})\,xy + \sqrt{z} - z^2 \\ \hline \end{array}$$

$$\begin{array}{l} 42 - a\sqrt{(x^2 - z^2)} + \sqrt[3]{m} \\ b\sqrt{(-z^2 + x^2)} + 10 - am^{\frac{1}{3}} \\ \hline (b - a)\,(x^2 - z^2)^{\frac{1}{2}} + (1 - a)\,m^{\frac{1}{3}} + 52 \\ \hline \end{array}$$

SUBTRACTION.

36. **Change** the signs of all the terms to be subtracted, and then collect the several quantities together as in Addition.

Thus, if $5a - 3a$ is to be subtracted from $4a$:
then $5a - 3a$ when the signs are changed
will be $-5a + 3a$, to this add $4a$ and
we have $-5a + 3a + 4a$
or $7a - 5a$ or $2a$, the required difference:

Therefore the truth of the rule is manifest, because $5a - 3a$, or $2a$ taken from $4a$ leaves $2a$.

37. **Again**, if $2b - 4b$ (or $-2b$) is to be taken from $3b$: then if the signs of $2b - 4b$ are changed, and $3b$ added, we have $-2b + 4b + 3b$, or $7b - 2b$, or $5b$ the required difference.

Therefore, subtracting $-2b$ is the same as adding $2b$: consequently subtracting a *negative* quantity gives the same result as adding an equal *positive* one.

38. When the quantities are alike, and have numeral coefficients, the operation may be performed as in common arithmetic, if those to be subtracted are the least:

$$\begin{array}{rl} \text{Thus, from} & -7x + 3z - 4y \\ \text{take} & -5x + z - 3y \\ \hline \text{rem.} & -2x + 2z - y \end{array}$$

Other Examples.

$$\begin{array}{rl} \text{From} & 2ax - bx + 3cx \\ \text{take} & -dx + 2fx - 3gx \\ \hline \text{diff.} & 2ax - bx + 3cx + dx - 2fx + 3gx \\ \text{or} & (2a + 3c + d + 3g - b - 2f)\,x. \end{array} \qquad \begin{array}{r} 5\sqrt{ax} - 3z^2 \\ -\sqrt{ax} + 2z^2 \\ \hline 6\sqrt{ax} - 5z^2 \end{array}$$

$$\begin{array}{rl} \text{From} & (ax + y)^2 + 5mx - \sqrt{(a^2 - x^2)} \\ \text{take} & 3a(ax + y)^2 + 6mx + a\sqrt{(a^2 - x^2)} \\ \hline \text{diff.} & (1 - 3a)(ax + y)^2 - mx - (1 + a)\sqrt{(a^2 - x^2)} \end{array}$$

By uniting the coefficients, as in the last example, the results are frequently simplified.

MULTIPLICATION.

39. The general rule for the signs in the product is, that like signs produce *plus* (+), and unlike signs *minus* (—).

Thus $+a \times +b$, or $a \times b$ gives $+ab$ or ab.
And $-a \times -b$ is also $+ab$ or ab.
But $+a \times -b$ produces $-ab$.

40. Simple factors, as a, b, c, &c. may be set down in any order to denote their product:

Thus $a \times b \times c$ is the same as abc or bca or cba, &c.
For suppose $a = 2$, $b = 3$, $c = 4$:
Then $abc = 24$ the continued product of 3, 4, and 2, any how varied.

41. When the factors have numeral coefficients, prefix their product to the letters with their proper signs:

Thus $3a \times 2b$ is $6ab$; this is manifest, if it be admitted that ab denotes the product of a and b:
for the factors of $3a \times 2b$ are 3, 2, a, and b,
and therefore $3 \times 2 \times a \times b$ is the same as $6 \times ab$, or $6ab$.

42. When a factor is multiplied into itself, the product becomes a power whose root is that factor:

Thus $a \times a \times a$ or aaa or a^3 is the third power or cube of a; where the small figure 3 is the index or exponent (11). And the cube root of a^3 is a.

And $aaaa$, &c. repeated to n times is a^n.

And the nth root of a^n is a.

Also $ab \times ab$ is $abab$ or $aabb$ or a^2b^2; therefore $(ab)^2$ is a^2b^2.
And $(cx)^n$ is c^nx^n.

Other Examples.

	$7a$	$9bc$	$-axy$
	$4b$	$-3d$	$-2bz$
Product	$28ab$	$-27bcd$	$+2abxyz$
	$-5x^2z$	$3z^3x$	$-\frac{1}{4}xyz$
	axz	$\frac{1}{2}ax^3$	$-\frac{2}{3}yzx$
Product	$-5ax^3z^2$	$1\frac{1}{2}az^3x^4$	$+\frac{1}{6}x^2y^2z^2$

43. When one of the factors is a compound quantity, the product is found by multiplying each of its terms by the other factor:

Thus $a + b$ multiplied by c is $ca + cb$,
And $a - b$ multiplied by c is $ca - cb$.

That $ca + cb$ is the product of the factors $a + b$ and c, will be manifest, if we consider that the whole is equal to all its parts taken together:

For let mc be the whole product of the factors c and m; then this whole (mc) is made up of several products, as $\frac{3}{4}mc + \frac{1}{4}mc$, or $\frac{1}{3}mc + \frac{1}{2}mc + \frac{1}{6}mc$, &c. &c.; therefore, whatever be the number of parts into which m is divided, the products of those parts, and the factor c, taken together, will be equal to the whole product mc. If, therefore, we consider a and b as the *parts* of a *whole*, the two products $ca + cb$ will denote the product of $a + b$ and c.

Let $a = 6$, $b = 2$, and $c = 3$:

Then $ca + cb = 18 + 6 = 24$, the same as the sum $6 + 2$ multiplied by 3.

Again, let $m - \frac{1}{4}m$, and c, be two factors; then their product will be equal to the difference of the products mc and $\frac{1}{4}mc$, or equal to $mc - \frac{1}{4}mc$:

For $mc - \frac{1}{4}mc = \frac{3}{4}mc$, which is the same as $(m - \frac{1}{4}m)c$, or $\frac{3}{4}m \times c$.

Here it is evident that instead of m and $\frac{1}{4}m$, we may make use of any other two quantities whose difference can be expressed in a simple term. It therefore appears that when a and b are quantities which can be compared, the product $(a - b)c$ is equal to $ca - cb$.

The same conclusion however, will be manifest, if we consider, that in order to have the product of $a - b$ and c, the product ca must be diminished by c times b, because a is greater than $a - b$ by b.

Hence $+c \times -b$ or $-b \times c$ is $-cb$, therefore, *unlike signs* give *minus* (—) in the product.

Corol. Therefore a compound expression, as $axyz - bxyz + 2cxyz$, where one of the factors (xyz) is common to all the terms, may be set down thus: $(a - b + 2c)\ xyz$.

44. When both the factors are compound quantities, multiply all the terms of the multiplicand by each of the terms in the multiplier; then collect the several products, as in Addition.

Thus, let $a - c$ be multiplied by $b - d$;

$$\begin{array}{rl} & a - c \\ & b - d \\ \hline \text{Product} & ab - bc - da + dc \\ \hline \end{array}$$

By the preceding article, the product of $a - c$ by b is $ab - bc$, which would be the true result, provided b was the only multiplier; but the multiplier is *less* than b by d, and therefore the product $ab - bc$ is d times $a - c$ *greater* than what ought to be produced by the multiplier $b - d$, consequently d times $a - c$ should be subtracted from $ab - bc$ to have the true product:

Now d times $a - c$ is $da - dc$, which subtracted from $ab - bc$, gives $ab - bc - da + dc$(36); the same result as by the rule. Therefore, $-c \times -d$ is $+ dc$; and consequently *like signs* give *plus* (+) in the product.

Other Examples.

$$\begin{array}{rlll} & 5xy - ab + z & -2\tfrac{1}{2}z + x & 3a^3 + 2x^3 - y^3 \\ & 4a & -14 & 3ax \\ \hline \text{Product} & 20axy - 4a^2b + 4az & 35z - 14x & 9a^4x + 6ax^4 - 3axy^3 \\ \hline \end{array}$$

$$\begin{array}{l} x + 1 \\ x + 1 \\ \hline x^2 + x \\ \quad + x + 1 \\ \hline x^2 + 2x + 1 \\ \hline \end{array}$$

the square of $x + 1$.

$$\begin{array}{l} x + y \\ x - y \\ \hline x^2 + xy \\ \quad - xy - y^2 \\ \hline x^2 \qquad - y^2 \\ \hline \end{array}$$

or $x^2 - y^2$, viz. the product of the sum and difference of two numbers is equal to the difference of their squares.

$$\begin{array}{l} x^3 + x^2y + xy^2 + y^3 \\ x - y \\ \hline x^4 + x^3y + x^2y^2 + xy^3 \\ \quad - x^3y - x^2y^2 - xy^3 - y^4 \\ \hline x^4 \qquad\qquad\qquad - y^4 \text{ or } x^4 - y^4 \\ \hline \end{array} \qquad \begin{array}{l} x^4 - x^3 + x^2 - x + 1 \\ x + 1 \\ \hline x^5 - x^4 + x^3 - x^2 + x \\ \quad + x^4 - x^3 + x^2 - x + 1 \\ \hline x^5 \qquad\qquad\qquad + 1 \text{ or } x^5 + 1. \\ \hline \end{array}$$

$$\begin{array}{ll} & x^2 - ax + b \\ & x + c \\ \cline{2-2} & x^3 - ax^2 + bx \\ & \quad + cx^2 - cax + cb \\ \cline{2-2} & x^3 + cx^2 - ax^2 + bx - cax + cb \\ \text{or,} & x^3 + (c-a)\,x^2 + (b-ca)\,x + cb \\ \text{or,} & x^3 - (a-c)\,x^2 - (ca-b)\,x + cb \end{array} \left.\vphantom{\begin{array}{l} x \\ x \end{array}}\right\} \text{by uniting the coefficients } a, c, b.$$

In this example let $a = 4$, $c = 2$, $b = 3$: then the two last expressions will be

$$x^3 + (2-4)\,x^2 + (3-8)\,x + 6, \text{ or } x^3 - 2x^2 - 5x + 6.$$
$$x^3 - (4-2)\,x^2 - (8-3)\,x + 6, \text{ or } x^3 - 2x^2 - 5x + 6.$$

But if c is greater than a, and b greater than ca, then x^2 and x with their coefficients will be affirmative.

45. Because $a \times a$ is a^2, $a \times a \times a$ is a^3, $a \times a \times a \times a \times a$ is a^5, &c. (42); it follows, that the addition of the indices answers to the multiplication of the factors (111. Arith.) for $4^2 \times a^3 = a^5 = a^{2+3}$, &c. Therefore, when powers of the same quantity are to be multiplied together, add the indices together for the index of the product:

$$\begin{array}{ll} \text{Thus } a \times a = a^{1+1} \text{ or } a^2 & \qquad a^2x^3 \times a^3x^2 = a^{2+3}x^{3+2} \text{ or } a^5x^5. \\ a^2 \times a = a^{2+1} \text{ or } a^3 & \qquad a^n \times a^m = a^{n+m} \\ a^3 \times a^3 = a^{3+3} \text{ or } a^6 & \qquad a^nb^m \times ab = a^{n+1}b^{m+1}: \end{array}$$

And $(a^2 - x^2)^n \times (a^2 - x^2)^m \times (a^2 - x^2)^r$ is $(a^2 - x^2)^{n+m+r}$

DIVISION.

46. DIVISION in Algebra, as in common Arithmetic, consists in finding a quantity which multiplied by the divisor, shall produce the dividend.

Therefore, the rule for the signs will be the same as in Multiplication; namely, like signs give *plus* in the quotient, and unlike signs *minus*.

Thus $+ab$ divided by $+b$ gives $+a$; for $a \times b$ is ab the dividend.

Also $-ab$ divided by $-b$ gives $+a$; because $-b \times a$ is $-ab$ the dividend.

But when ab is divided by $-b$, the quotient will be $-a$; for $-b \times -a$ is $+ab$ or ab.

47. When the divisor and dividend are simple quantities, the quotient, in most cases, may be discovered by inspection only, if we make a fraction of the terms, and consider it as the result of the division (37. Arith.)

Thus if 3 be divided by 9, the quotient is $\frac{3}{9}$ or $\frac{1}{3}$:

In like manner when $3abc$ is divided by bc, the quotient may be denoted by $\frac{3abc}{bc}$:

And dividing the numerator and denominator by the factors (bc), which are common to both, we have $\frac{3abc}{bc} = \frac{3a}{1} = 3a$ the quotient:

For bc (the divisor) $\times\ 3a$ gives $3abc$ the dividend.

Other Examples.

Divide $15axy$ by $-3ay$

$\frac{15axy}{-3ay} = -5x$ the quotient: for $-3ay \times -5x = 15axy$.

Divide $-7a^2z^2$ by $-14az$.

$\frac{-7a^2z^2}{-14az} = \frac{1}{2}az$ the quotient: for $-14az \times \frac{1}{2}az = -7a^2z^2$.

Divide $-a^3z^3$ by $5az^2$.

$\frac{-a^3z^3}{5az^2} = -\frac{1}{5}a^2z$ the quotient: for $-\frac{1}{5}a^2z \times 5az^2 = -a^3z^3$.

Divide $4axy$ by $4bxy$.

$\frac{4axy}{4bxy} = \frac{a}{b}$ the quotient.

Divide $-5zx$ by $-10azx$.

$\frac{-5zx}{-10azx} = \frac{1}{2a}$ the quotient.

The preceding operations are evidently analogous to that of reducing Fractions to their lowest terms in Arithmetic.—(39. Arith.)

48. Divide $3xy$ by $5az$. Here the divisor and dividend have no common factor, and therefore the quotient is $\frac{3xy}{5az}$.

49. When powers of the same quantity are to be divided one by the other, subtract the index of the divisor from that of the dividend, and the difference will be the index of the quotient.

Thus a^5 divided by a^2 gives a^{5-2} or a^3 the quotient;

For $a^2 \times a^3 = a^5$ (45).

Or denoting the quotient by the fraction $\frac{a^5}{a^2}$, and reducing it to its lowest terms,

$$\frac{a^5}{a^2} = \frac{a^2 \times a^3}{a^2} = \frac{a^3}{1} = a^3 \text{ the quotient.}$$

Also, $x^4 \div x^3$ is $x^{4-3} = x^1 = x$ the quotient.

And x^m divided by x^3 is x^{m-3}.

$x^m \div x^n$ is x^{m-n}.

Also $(c-z)^{m} \div (c-z)^{\frac{1}{r}}$ or $\frac{(c-z)^{m}}{(c-z)^{\frac{1}{r}}}$, is $(c-z)^{m-\frac{1}{r}}$.

50. **When the index of the divisor is greater than that of the dividend, the quotient will have a negative index.**

Thus x^3 divided by x^5 will give x^{3-5} or x^{-2}; for $3-5$ is -2, (36. 31.)

But $x^3 \div x^5$ may be denoted by $\frac{x^3}{x^5}$ which reduced to its lowest terms is $\frac{1}{x^2}$ therefore $\frac{1}{x^2}$ is the same as x^{-2}.

51. **If the dividend be a compound quantity, and the divisor a simple one, each term of the former must be divided by the latter, as in the foregoing examples.**

Thus, let $ab-ac$ be divided by a:

Then the quotient may be set down thus $\frac{ab-ac}{a}$, which reduced is $\frac{b-c}{1}$ or $b-c$, the quotient.

For $(b-c)\ a$ is $= ab-ac$.

Or the quotient may be denoted thus $\frac{ab}{a}-\frac{ac}{a}$, and these fractions reduced are $\frac{b}{1}-\frac{c}{1}$ or $b-c$ the quotient, as before.

Other Examples.

Divide $6acx-8a^2c-10a^2cx$ by $2ac$.

$$\frac{6acx-8a^2c-10a^2cx}{2ac}=3x-4a-5ax \text{ the quotient.}$$

Divide $5z^2-14ax+16z^3$ by 7.

$$\frac{5z^2}{7}-\frac{14ax}{7}+\frac{16z^3}{7}, \text{ or } \frac{5}{7}z^2-2ax+2\tfrac{2}{7}z^3 \text{ the quotient.}$$

Divide $9ax^2 - 12a^3x^2$ by $-3ax$.

$$\frac{9ax^2 - 12a^3x^2}{-3ax} = -3x + 4a^2x \text{ the quotient.}$$

Divide $acx + cx - ac$ by ac.

$\frac{acx + cx - ac}{ac} = \frac{ax + x - a}{a}$, the quotient: this is found by cancelling c, the only factor common to the whole numerator, and denominator: but if we denote the quotient by three fractions, it may be reduced to a more simple form:

Thus $\frac{acx}{ac} + \frac{cx}{ac} - \frac{ac}{ac} = x + \frac{x}{a} - 1$, the quotient. This, however, is only the former quotient reduced: for $\frac{ax + x - a}{a}$ is the same as $\frac{ax}{a} + \frac{x}{a} - \frac{a}{a}$, or $x + \frac{x}{a} - 1$.

Divide $(x^2 + z)^2 + ay$ by bz.

Here the divisor and dividend have no common factor, and, therefore, $\frac{(x^2 + z)^2 + ay}{bz}$ is the quotient.

52. When the divisor and dividend are both compound quantities; Arrange their terms according to the powers of some one letter in both, the higher powers being to the left.

Find how often the first term of the divisor is contained in the first term of the dividend, and set the result in the quotient.

Multiply the whole divisor by the result thus found, and subtract the product from the dividend: to this remainder bring down as many other terms of the dividend as are necessary for the next operation; then divide as before, and so on, till all the terms are brought down.

Thus to divide $a^2 - 2ab + b^2$ by $a - b$.

$$\begin{array}{rl}
a - b) & a^2 - 2ab + b^2 \ (a - b \text{ quotient.} \\
 & \underline{a^2 - ab} \\
 & \quad -ab + b^2 \\
 & \quad \underline{-ab + b^2} \\
 & \qquad\quad 0
\end{array}$$

Here a, the left hand term of the divisor, is contained a times in a^2, the left hand term of the dividend, therefore a is the first term of the quotient; and the divisor $a-b$ multiplied by a is a^2-ab, which taken from a^2-2ab in the dividend leaves $-ab$; to this bring down $+b^2$, and $-ab+b^2$ is the second dividend.

Next, a in the divisor is contained $-b$ times in $-ab$ (the left hand term of the dividend $-ab+b^2$); therefore $-b$ is the second term in the quotient: now the divisor $a-b$ multiplied by $-b$ gives $-ab+b^2$ the second dividend: therefore $a-b$ is the quotient without a remainder.

For $(a-b)\times(a-b)=a^2-2ab+b^2$; the proof, as in common arithmetic.

Other Examples.

Divide x^5+1 by $x+1$.

$$\begin{array}{l}
x+1)\ x^5+1\ (x^4-x^3+x^2-x+1 \text{ quotient.}\\
\qquad\quad \underline{x^5+x^4}\\
\qquad\qquad -x^4+1\\
\qquad\qquad \underline{-x^4-x^3}\\
\qquad\qquad\qquad +x^3+1\\
\qquad\qquad\qquad \underline{+x^3+x^2}\\
\qquad\qquad\qquad\qquad -x^2+1\\
\qquad\qquad\qquad\qquad \underline{-x^2-x}\\
\qquad\qquad\qquad\qquad\qquad +x+1\\
\qquad\qquad\qquad\qquad\qquad \underline{+x+1}\\
\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}$$

In this example, x is contained x^4 times in x^5, and the divisor $x+1$ multiplied by x^4 gives x^5+x^4, which subtracted from x^5+1 the dividend and $-x^4+1$ remains, the second dividend.

Next, x is contained $-x^3$ times in $-x^4$, therefore $-x^3$ is the second term in the quotient; and $(x+1)\times -x^3$ gives $-x^4-x^3$, which taken from $-x^4+1$, and the remainder is $+x^3+1$. And so on.

$(x^4-x^3+x^2-x+1)\times(x+1)=x^5+1$. See the multiplication, Art. 44.

Divide $6x^2-x-12$ by $2x-3$.

$$\begin{array}{l}
2x-3)\ 6x^2-x-12\ (3x+4 \text{ quotient,}\\
\qquad\quad \underline{6x^2-9x}\\
\qquad\qquad +8x-12\\
\qquad\qquad \underline{+8x-12}\\
\qquad\qquad\quad 0
\end{array}$$

$$\begin{array}{l}
x+c)\;x^3+cx^2-ax^2-cax+bx+cb\;(x^2-ax+b \text{ quotient.}\\
\qquad\underline{x^3+cx^2}\\
\qquad\qquad\quad -ax^2-cax\\
\qquad\qquad\quad \underline{-ax^2-cax}\\
\qquad\qquad\qquad\qquad +bx+cb\\
\qquad\qquad\qquad\qquad \underline{+bx+cb}\\
\qquad\qquad\qquad\qquad\quad 0
\end{array}$$

$$\begin{array}{l}
x+z)\;x^2-z^2+y\;\left(x-z+\dfrac{y}{x+z} \text{ the quotient.}\right.\\
\qquad\underline{x^2+xz}\\
\qquad\quad -xz-z^2\\
\qquad\quad \underline{-xz-z^2}\\
\qquad\qquad\quad +y
\end{array}$$

Here y is the remainder, therefore $\dfrac{y}{x+z}$ is a fractional part of the quotient, as in the division of whole numbers. In these cases however, the quotient is usually set down thus, $\dfrac{x^2-z^2+y}{x+z}$.

$$\begin{array}{l}
2x^2-3ax+a^2)\;4x^4-9a^2x^2+6a^3x-a^4\;(2x^2+3ax-a^2 \text{ quotient.}\\
\qquad\qquad\qquad \underline{4x^4-6ax^3+2a^2x^2}\\
\qquad\qquad\qquad\quad +6ax^3-11a^2x^2+6a^3x\\
\qquad\qquad\qquad\quad \underline{+6ax^3-9a^2x^2+3a^3x}\\
\qquad\qquad\qquad\qquad\qquad -2a^2x^2+3a^3x-a^4\\
\qquad\qquad\qquad\qquad\qquad \underline{-2a^2x^2+3a^3x-a^4}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}$$

$$\begin{array}{l}
x+y)\;x^2-y^2\;(x-y \text{ quotient.}\\
\qquad\underline{x^2+xy}\\
\qquad\quad -xy-y^2\\
\qquad\quad \underline{-xy-y^2}\\
\qquad\qquad\quad 0
\end{array}$$

53. By the last example it appears, that the difference of two squares is divisible by the sum, and also by the difference of their roots.

Again, $\dfrac{x^3+y^3}{x+y}=x^2-xy+y^2$ the quotient.

$\dfrac{x^3-y^3}{x-y}=x^2+xy+y^2$ quotient.

$\dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3$ quotient,

$$\frac{x^4 - y^4}{x - y} = x^3 + x^2y + xy^2 + y^3 \text{ quotient.}$$

$$\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4 \text{ quotient.}$$

$$\frac{x^5 - y^5}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4 \text{ quotient.}$$

&c. &c.

54. Hence we conclude, that $x^n + y^n$ is divisible by the sum of the roots $x + y$, and $x^n - y^n$ by their difference, when the index n is an odd number:

And that $x^n - y^n$ is divisible by the sum, and also by the difference when n is an even number.

ALGEBRAIC FRACTIONS.

55. The learner should perfectly understand the theory and practice of Vulgar Fractions in Arithmetic, before he attempts this part of Algebra, because he will perceive that the same rules answer equally in both.

To reduce a Fraction to its lowest Terms.

56. Divide the numerator and denominator by the factor or factors common to both, and the result will be the answer.

The rule is derived from this obvious principle, that, if the terms of a fraction are multiplied, or divided by the same number or quantity, its value is not altered.

Thus, let $\frac{6a}{8b}$ be the fraction:

then $\frac{6a}{8b} = \frac{2 \times 3a}{2 \times 4b}$, therefore 2 is a factor common to the numerator and denominator, and because it is the greatest common measure of $6a$ and $8b$, the fraction in its lowest terms is $\frac{3a}{4b}$.

And $\frac{ab}{ac}$ reduced is $\frac{b}{c}$; for a is the factor common to both terms.

Also $\frac{-5a^2xy}{10axz}$ reduced is $\frac{-ay}{2z}$; where $5ax$ is the common factor.
For $\frac{-ay \times 5ax}{2z \times 5ax} = \frac{-5a^2xy}{10\,axz}$.

Again, $\frac{4abz - 7aby}{2abz + 6abx}$ reduced is $\frac{2z - 3\frac{1}{2}y}{z + 3x}$.

57. **The simple divisors are readily found, as in the preceding examples. But to discover the compound divisors, let the terms of the fraction be resolved into their factors :**

Thus, to reduce the fraction $\frac{4a^2x^2 - 4a^2y^2}{bx + by}$ to its lowest terms :

$$\frac{4a^2x^2 - 4a^2y^2}{bx + by} = \frac{4a^2\,(x^2 - y^2)}{b\,(x + y)} = \frac{4a^2\,(x - y)\,(x + y)}{b\,(x + y)}$$

therefore both terms of the fraction are divisible by $x + y$, which is evidently their greatest common measure; and the fraction in its lowest terms is $\frac{4a^2\,(x - y)}{b}$ or $\frac{4a^2x - 4a^2y}{b}$.

58. **But when the numerator and denominator consist of several terms, the usual method of proceeding is to find their greatest common measure thus :**

Reject the simple divisors in both terms of the fraction, then,

Divide the greater by the less, and the last divisor by the last remainder, and so on till nothing remains; then the last divisor is the greatest common measure, as in Arithmetic. (40. Arith.)

Thus, to reduce $\frac{5a^5 + 10a^4b + 5a^3b^2}{a^3b + 2a^2b^2 + 2ab^3 + b^4}$ to its lowest terms:

Then $\frac{5a^5 + 10a^4b + 5a^3b^2}{a^3b + 2a^2b^2 + 2ab^3 + b^4} = \frac{(a^2 + 2ab + b^2)\;5a^3}{(a^3 + 2a^2b + 2ab^2 + b^3)\;b}$;
for the simple divisors are $5a^3$ and b:

Therefore we have to find the greatest common measure of $a^2+2ab+b^2$ and $a^3+2a^2b+2ab^2+b^3$:

$$\begin{array}{rl}
a^2+2ab+b^2) & a^3+2a^2b+2ab^2+b^3\ (a \\
 & \underline{a^3+2a^2b+ab^2} \\
 & \qquad ab^2+b^3)\ a^2+2ab+b^2 \\
 & \qquad \text{or}\ \ a\ +b)\ a^2+2ab+b^2\ (a+b. \\
 & \qquad\qquad\qquad \underline{a^2+ab} \\
 & \qquad\qquad\qquad\quad ab+b^2 \\
 & \qquad\qquad\qquad\quad \underline{ab+b^2} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}$$

The first remainder is ab^2+b^3, therefore the next operation is that of finding the greatest common measure of ab^2+b^3, and the last divisor $a^2+2ab+b^2$, for which reason we reject b^2 the simple divisor of ab^2+b^3, and then it is reduced to finding the greatest common measure of $a+b$ and $a^2+2ab+b^2$, which is $a+b$ the last divisor.

$a+b\Big)\dfrac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}\Big(=\dfrac{5a^4+5a^3b}{a^2b+ab^2+b^3}$ the fraction in its lowest terms.

The reason that $a+b$ measures the terms of the proposed fraction is evident from this consideration, that if a divisor measures a quantity, it must also measure any multiple of that quantity.

To reduce $\dfrac{8a^2b-10ab^2+2b^3}{9a^4-9a^3b+3a^2b^2-3ab^3}$ to its lowest terms.

The simple divisors are $2b$ and $3a$, hence the fraction becomes

$$\frac{(4a^2-5ab+b^2)\,2b}{(3a^3-3a^2b+ab^2-b^3)\,3a}:$$

Therefore we have to find the greatest common measure of the terms between the parentheses:

Now, that the less may divide the greater, let the latter be multiplied by 4:

$$\begin{array}{rl}
 & 3a^3-3a^2b+ab^2-b^3 \\
 & \underline{4\qquad\qquad\qquad\qquad\qquad} \\
4a^2-5ab+b^2) & 12a^3-12a^2b+4ab^2-4b^3\ (3a \\
 & \underline{12a^3-15a^2b+3ab^2} \\
 & \qquad\quad +3a^2b+ab^2-4b^3 \\
\text{again, multiply by}\ldots\ldots\ldots\ldots\ldots & \qquad\quad \underline{4\qquad\qquad\qquad\qquad} \\
 & \qquad\quad +12a^2b+4ab^2\quad 16b^3\ (3b \\
 & \qquad\quad \underline{+12a^2b}-15ab^2+3b^3 \\
\text{divide by } 19b^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \qquad\qquad\qquad +19ab^2-19b^3 \\
 & \qquad\qquad\qquad\quad +a-b)
\end{array}$$

$$\begin{array}{r} +a-b)\ 4a^2-5ab+b^2\ (4a-b \\ \underline{4a^2-4ab} \\ -ab+b^2 \\ \underline{-ab+b^2} \\ 0 \end{array}$$

therefore the last divisor $a-b$ is the greatest common measure.

$a-b\Big)\dfrac{8a^2b-10ab^2+2b^3}{9a^4-9a^3b+3a^2b^2-3ab^3}\Big(\dfrac{8ab-2b^2}{9a^3+3ab^2}$ the fraction in its lowest terms.

59. The multiplication of the dividends (as in the last example) cannot affect the common measure, because the dividends thus increased, are only multiples of the former dividends: Sometimes however, the necessary factor or factors for that purpose, are not discoverable at first sight; for example,

Let it be required to reduce the fraction $\dfrac{3bcz+5mxz+30mx+18bc}{4adz-7vrz+24ad-42vr}$ to its lowest terms:

Here it appears that a numeral multiplier will not answer the purpose, and therefore one term of the fraction must be multiplied by some factor or factors of the other before a division can take place: but in the present case, the shortest method of reduction is that of resolving the numerator and denominator into their factors:

thus $\dfrac{3bcz+5mxz+30mx+18bc}{4adz-7vrz+24ad-42vr}=\dfrac{(3bc+5mx)z+30mx+18bc}{(4ad-7vr)z+24ad-42vr}$

$=\dfrac{(3bc+5mx)z+(5mx+3bc)6}{(4ad-7vr)z+(4ad-7vr)6}=\dfrac{(3bc+5mx)(z+6)}{(4ad-7vr)(z+6)}$; therefore $z+6$ is a common divisor; and the fraction is reduced to $\dfrac{3bc+5mx}{4ad-7vr}$, which is in its lowest terms.

To reduce an improper Fraction to its equivalent whole or mixed quantity.

60. This is nothing more than a division; therefore, divide the numerator by the denominator, and the quotient will be the answer. (53.)

Thus, if the fraction to be reduced is $\frac{6ab^2}{2ab}$:

Then $\frac{6ab^2}{2ab} = 3b$ the quotient, or the fraction reduced to its equivalent *whole.* Consequently when there is no remainder after division, the operation is that of reducing a fraction to its lowest terms; for $\frac{6ab^2}{2ab} = \frac{3b}{1}$, the fraction in its lowest terms.

Also $\frac{x^3+y^3}{x+y}$ reduced is $x^2 - xy + y^2$. (53.)

Suppose the fraction to be reduced is $\frac{3ac-4bc}{a-b}$:

$$
\begin{array}{l}
a-b)\ 3ac-4bc\ \left(3c+\frac{-bc}{a-b}, \text{ or } 3c-\frac{bc}{a-b}, \text{ the quotient.}\right. \\
\qquad \underline{3ac-3bc} \\
\qquad\qquad \underline{-bc}
\end{array}
$$

Here the remainder is $-bc$, therefore $\frac{-bc}{a-b}$ is the fractional part of the required mixed quantity, but this fraction is negative (46); and since $+\frac{-bc}{a-b}$ and $-\frac{bc}{a-b}$ denote the same thing, the quotient may be set down either way.

Again, let the proposed fraction be $\frac{ac-cb-m-n}{a-b}$.

$$
\begin{array}{l}
a-b)\ ac-cb-m-n\ \left(c+\frac{-m-n}{a-b}, \text{ or } c-\frac{m+n}{a-b}, \text{ the quotient.}\right. \\
\qquad \underline{ac-cb} \\
\qquad\qquad \underline{-m-n}
\end{array}
$$

In this example the remainder is $-m-n$, and the fraction is $\frac{-m-n}{a-b}$; but $-m-n$ is the same as $-(m+n)$, namely, the sum of m and n is negative; therefore $+\frac{-m-n}{a-b}$ and $-\frac{m+n}{a-b}$ are expressions for the same quantity.

To reduce a mixed quantity to an equivalent Fraction.

61. THIS is the reverse of the operation in the preceding article; therefore,

Multiply the integral part by the denominator of the fraction, and add its numerator to the product; then the sum placed over the said denominator will form the fraction required.

Thus, let the mixed quantity be $a+\frac{x-y}{a+b}$.

$$\begin{array}{rl} \text{then } a\times(a+b) = & a^2+ab \\ \text{add} \ldots\ldots & x-y \\ \hline \text{sum} & a^2+ab+x-y \end{array}$$

and $\frac{a^2+ab+x-y}{a+b}$ is the fraction sought.

Reduce $3c-\frac{bc}{a-b}$ to an equivalent fraction.

$$\begin{array}{rl} 3c\times(a-b) = & 3ac-3cb \\ & \quad\;\; -bc \text{ add} \\ \hline & 3ac-4bc \text{ the sum:} \end{array}$$

therefore the fraction is $\frac{3ac-4bc}{a-b}$.

Reduce $c+\frac{-m-n}{a-b}$ $\left(\text{or } c-\frac{m+n}{a-b}\right)$ to an equivalent fraction:

$$\begin{array}{rl} c\times(a-b) = & ac-bc \\ & -m-n \text{ add} \\ \hline & ac-bc-m-n \text{ the sum,} \end{array}$$

and $\frac{ac-bc-m-n}{a-b}$ is the fraction required.

If the quantity were given in this form, $c-\frac{m+n}{a-b}$, the thing to be done is evidently that of subtracting the fraction $\frac{m+n}{a-b}$ from the integer c.

Also, $cx-ax-\frac{b-c}{-a}$ reduced to a fraction is $\frac{-cax+a^2x-b+c}{-a}$, which denotes the difference of $cx-ax$ and $\frac{b-c}{-a}$.

To bring Fractions with different denominators to equivalent Fractions having a common denominator.

62. MULTIPLY each numerator into all the denominators, except its own, for the new numerator of that fraction; and all the denominators together for the common denominator.

The rule may be investigated exactly as in Arithmetic. (45. Arith.)

Let the fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{m}{n}$, be brought to equivalent fractions having a common denominator.

$$\left.\begin{matrix} adn \\ cbn \\ mdb \end{matrix}\right\}$$ the three new numerators.

bdn the common denominator.

And the three fractions are $\frac{adn}{bdn}$, $\frac{cbn}{bdn}$, $\frac{mdb}{bdn}$.

Reduce $\frac{a}{b}$, $\frac{b}{c}$, $\frac{a}{c}$, to a common denominator.

$$\left.\begin{matrix} acc \\ bbc \\ acb \end{matrix}\right\}$$ the numerators.

bcc the common denominator.

Hence the fractions are $\frac{acc}{bcc}$, $\frac{bbc}{bcc}$, $\frac{acb}{bcc}$, or $\frac{ac}{bc}$, $\frac{bb}{bc}$, $\frac{ab}{bc}$, in their lowest terms.

63. When the denominator of one fraction is a multiple of the denominator of another, divide the greater denominator by the less, then multiply the terms of that fraction which hath the least denominator by the quotient, and the two fractions will be reduced to a common denominator. (47. Arith.)

Thus, if the fractions are $\frac{m}{a}$ and $\frac{2x}{ab}$:

Then ab is a multiple of a; and ab divided by a gives the quotient b; therefore if the terms of the fraction $\frac{m}{a}$ are multiplied by b, we have $\frac{mb}{ab}$ which has the same denominator as $\frac{2x}{ab}$.

In like manner the fractions $\frac{x}{a}$, $\frac{2z}{ac}$, $\frac{y}{acd}$, when brought to a common denominator are $\frac{cdx}{acd}$, $\frac{2dz}{acd}$, $\frac{v}{acd}$: For the terms of the first $\left(\frac{x}{a}\right)$ are multiplied by cd, and those of the second by d.

64. Let $a+x$ and $\frac{nx}{ac}$ be brought to fractions having a common denominator.

Making 1 the denominator of $a+x$ gives the fraction $\frac{a+x}{1}$, and if its terms are multiplied by ac we get $\frac{a^2c+acx}{ac}$: and the two required fractions are $\frac{a^2c+acx}{ac}$ and $\frac{nx}{ac}$.

Hence, an integral quantity $(a+x)$ is brought to an equivalent fraction, having a given denominator (ac), by multiplying the former by the latter, and placing the product over that denominator.

To Add fractional quantities together.

65. BRING the fractions to a common denominator, then add the numerators together, and place the sum over the common denominator, as in Vulgar Fractions.

Examples.

1. Required the sum of the fractions $\frac{ax}{b}$, $\frac{cx}{b}$, and $\frac{dx}{b}$?

$$\frac{ax}{b}+\frac{cx}{b}+\frac{dx}{b}=\frac{ax+cx+dx}{b} \text{ or } \frac{(a+c+d)x}{b} \text{ the sum.}$$

2. Required the sum of $\frac{5x}{9}$, $\frac{5x}{6}$, and $\frac{-3x}{4}$?

The fractions brought to a common denominator are

$$\frac{20z}{36},\ \frac{30z}{36},\ \frac{-27z}{36}.$$

therefore $\frac{20z}{36}+\frac{30z}{36}-\frac{27z}{36}=\frac{20z+30z-27z}{36}=\frac{23z}{36}$ the answer.

Or the fractions $\frac{5}{9}$, $\frac{5}{6}$, and $\frac{3}{4}$ may be considered as the coefficients of z;

then $\frac{5}{9}, +\frac{5}{6}-\frac{3}{4}=\frac{23}{36}$, and the answer is $\frac{23}{36}z$, as before.

3. Let $\frac{a}{a-c}$ and $\frac{a-c}{a}$ be added together.

$$\frac{a}{a-c}=\frac{a^2}{a^2-ac},\ \text{and}\ \frac{a-c}{a}=\frac{a^2-2ac+c^2}{a^2-ac}:$$

then $\frac{a^2}{a^2-ac}+\frac{a^2-2ac+c^2}{a^2-ac}=\frac{2a^2-2ac+c^2}{a^2-ac}=2+\frac{c^2}{a^2-ac}$ the sum.

4. Suppose the fractions to be added together are $\frac{a-b}{c+d}$ and $\frac{a+b}{c-d}$.

Reducing the fractions to a common denominator, they take this form,

$$\frac{(a-b)\ (c-d)}{(c+d)\ (c-d)}\ \text{and}\ \frac{(c+d)\ (a+b)}{(c+d)\ (c-d)},$$

Then $\frac{(a-b)\ (c-d)+(c+d)\ (a+b)}{(c+d)\ (c-d)}=\frac{2ac+2bd}{c^2-d^2}$ the sum required.

5. To add $\frac{x-y}{2}$, $\frac{x-y}{5}$, and $\frac{13x-13y}{10}$ together

The three fractions may be denoted thus, $\frac{1}{2}(x-y)$, $\frac{1}{5}(x-y)$, and $\frac{13}{10}(x-y)$.

Then $\frac{1}{2}(x-y)+\frac{1}{5}(x-y)+\frac{13}{10}(x-y)$, or $(\frac{1}{2}+\frac{1}{5}+\frac{13}{10})(x-y)=(\frac{20}{10})(x-y)=2x-2y$ the sum of the three fractions.

To Subtract one fractional quantity from another.

66. BRING the fractions to a common denominator; then set the difference of the numerators over the common denominator for the answer, as in Vulgar Fractions.

Examples.

1. From $\frac{5ax}{16}$ take $\frac{3ax}{16}$?

$$\frac{5ax - 3ax}{16} = \frac{2ax}{16} = \frac{ax}{8} \text{ the remainder.}$$

2. From $\frac{dx}{a-c}$ take $\frac{dx}{a+c}$?

The fractions with a common denominator are $\frac{dx\,(a+c)}{a^2-c^2}$ and $\frac{dx\,(a-c)}{a^2-c^2}$;

then $\frac{dx\,(a+c) - dx\,(a-c)}{a^2-c^2} = \frac{dx\,(2c)}{a^2-c^2}$ the required difference.

3. Let $\frac{a-z}{a\,(b-z)}$ be subtracted from $\frac{c+z}{d\,(b-z)}$?

$\frac{d\,(a-z)}{da\,(b-z)}$ and $\frac{a\,(c+z)}{da\,(b-z)}$ are the fractions with a common denominator.

Therefore $\frac{a\,(c+z) - d\,(a-z)}{da\,(b-z)} = \frac{ac - da + z\,(a+d)}{da\,(b-z)}$ the remainder.

4. Let the fraction $\frac{m+n}{a-b}$ be subtracted from the integer c. (61.)

The integer c brought to an equivalent fraction having the denominator $a-b$, is $\frac{ca-cb}{a-b}$ (64).

Then $\frac{ca-cb}{a-b} - \frac{m+n}{a-b} = \frac{ca-cb-m-n}{a-b}$ is the fraction denoting the difference.

To Multiply fractional quantities together.

67. MULTIPLY the numerators together for the numerator of the product, and the denominators together for its denominator, as in Vulgar Fractions.

Examples.

1. Required the product of $\frac{a}{b}$, $\frac{d}{c}$, and $\frac{m}{n}$?

$\frac{a \times d \times m}{b \times c \times n}$ or $\frac{adm}{bcn}$ the product required.

2. What is the product of $\frac{3x}{4}$, $\frac{3x-3}{5}$, and $\frac{10}{9}$?

$$\frac{3x\ (3x-3)\ 10}{4\times 5\times 9}=\frac{90x^2-90x}{180}=\tfrac{1}{2}x^2-\tfrac{1}{2}x \text{ the product.}$$

3. What is the product of $\frac{ax}{by}$ and $\frac{b^2y^2}{ax^2}$?

$$\frac{ax\times b^2y^2}{by\times ax^2} \text{ or } \frac{ax\times by\times by}{by\times ax\times x}=\frac{by}{x} \text{ the product in its lowest terms.}$$

68. By resolving the terms of the fractions into their factors, the operation is frequently abridged:

4. Thus, to find the product of $\frac{a^2-x^2}{a+b}$, $\frac{ma^2-mb^2}{ax+x^2}$, and $\frac{bx^3}{mna-mnb}$?

$$\frac{a^2-x^2}{a+b}=\frac{(a+x)\ (a-x)}{a+b},$$

$$\frac{ma^2-mb^2}{ax+x^2}=\frac{m\ (a+b)\ (a-b)}{x\ (a+x)}.$$

$$\frac{bx^3}{mna-mnb}=\frac{bx^2\times x}{mn\ (a-b)}.$$

Then, $\frac{(a+x)\ (a-x)\times m\ (a+b)\ (a-b)\times bx^2\times x}{(a+b)\times x\ (a+x)\times mn\ (a-b)}=\frac{(a-x)\times bx^2}{n}$ $=\frac{abx^2-bx^3}{n}$, the product, by rejecting the like factors in the numerator and denominator, as in reducing fractions to their lowest terms. (56.)

69. The product of an integral and fraction is found by multiplying the numerator of the fraction by the integral, as in Vulgar Fractions.

$$\text{Thus } (a+x)\times x\times\frac{5}{x-4}=\frac{5ax+5x^2}{x-4} \text{ the product.}$$

70. Powers of the same fraction are multiplied together by the addition of their exponents, in the same manner as integral quantities. (45.)

$$\text{Thus } \left(\frac{b}{a}\right)^m\times\left(\frac{b}{a}\right)^n=\left(\frac{b}{a}\right)^{m+n}, \text{ the product.}$$

And $\frac{x^m}{c\,(a+x)^r} \times \frac{ax^{-n}}{b\,(a+x)^v} = \frac{ax^{m-n}}{cb\,(a+x)^{r+v}}$.

Also $\left(\frac{xz}{y}\right)^{m-1} \times \frac{xz}{y} = \left(\frac{xz}{y}\right)^m$ the product.

Division of fractional quantities.

71. INVERT the divisor, then proceed as in Multiplication. This rule is the same as that for Vulgar Fractions in Arithmetic.

Examples.

1. Divide $\frac{9ax}{16by}$ by $\frac{3a}{4b}$?

$\frac{4b}{3a} \times \frac{9ax}{16by}$, or $\frac{4b \times 3a \times 3x}{3a \times 4b \times 4y} = \frac{3x}{4y}$ the quotient (by rejecting the like factors in the numerator and denominator).

Or thus, $\left.\frac{3a}{4b}\right) \frac{9ax}{16by} \left(\frac{3x}{4y}\right.$ the quotient, as before.

2. Let $\frac{2ax+x^2}{c^3-x^3}$ be divided by $\frac{x}{c-x}$?

$\frac{2ax+x^2}{c^3-x^3} \times \frac{c-x}{x} = \frac{c-x}{c^3-x^3} \times \frac{(2a+x)\,x}{x} = \frac{c-x}{c^3-x^3} \times \frac{2a+x}{1}$ the quotient.

Now $\frac{c-x}{c^3-x^3}$ in its lowest terms is $\frac{1}{x^2+cx+c^2}$:

therefore $\frac{1}{x^2+cx+c^2} \times \frac{2a+x}{1} = \frac{2a+x}{x^2+cx+c^2}$ is the quotient in its lowest terms.

3. Divide $x + \frac{2x}{x-3}$ by $x - \frac{2x}{x-3}$?

$$x + \frac{2x}{x-3} = \frac{x^2-x}{x-3}; \text{ and } x - \frac{2x}{x-3} = \frac{x^2-5x}{x-3}.$$

Then $\frac{x^2-x}{x-3} \times \frac{x-3}{x^2-5x} = \frac{x-1}{1} \times \frac{1}{x-5} = \frac{x-1}{x-5}$ the quotient.

4. Let $\frac{3a\,(c-z)^{\frac{1}{3}}}{2x-2z}$ be divided by $\frac{5x\,(c-z)^{\frac{1}{3}}}{2x+2z}$.

$$\frac{3a\,(c-z)^{\frac{1}{3}}}{2x-2z} \times \frac{2x+2z}{5x\,(c-z)^{\frac{1}{3}}} = \frac{3a\ (x+z)}{5x\ (x-z)}$$ the quotient.

5. Divide $\frac{az^2}{c}$ by $\frac{dz^5}{b}$?

$\frac{az^2}{c} \times \frac{b}{dz^5} = \frac{ab}{cd} \times \frac{z^2}{z^5} = \frac{ab}{cd} \times \frac{1}{z^3}$ the quotient.

But $\frac{z^2}{z^5} = z^{2-5} = z^{-3}$ (50); therefore $\frac{ab}{cd} \times z^{-3}$ also denotes the quotient.

6. Let $\frac{a^{-m}}{b^{-m}}$ be divided by $\frac{a^m}{b^m}$.

$$\frac{a^m}{b^m}\Big)\ \frac{a^{-m}}{b^{-m}}\ \Big(\frac{a^{-2m}}{b^{-2m}}$$ the quotient. (49).

But $a^{-2m} = \frac{1}{a^{2m}}$, and $b^{-2m} = \frac{1}{b^{2m}}$, (50).

Therefore $\frac{a^{-2m}}{b^{-2m}} = \frac{b^{2m}}{a^{2m}}$, quotient.

To change a fractional quantity into a Series.

72. **Divide** the numerator by the denominator, and extend the quotient to as many terms as may be thought necessary.

Examples.

1. Let the fraction $\frac{a}{a+x}$ be changed to a series.

$$a+x\Big)\ a\ \Big(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+,\ \&c.$$

$$\begin{array}{l}
\underline{a+x} \\
\quad -x \\
\quad \underline{-x-\frac{x^2}{a}} \\
\qquad +\frac{x^2}{a} \\
\qquad \underline{+\frac{x^2}{a}+\frac{x^3}{a^2}} \\
\qquad\qquad -\frac{x^3}{a^2} \\
\qquad\qquad \underline{-\frac{x^3}{a^2}-\frac{x^4}{a^3}} \\
\qquad\qquad\qquad +\frac{x^4}{a^3}
\end{array}$$

Now it is easy to perceive, that the next or 5*th* term of the quotient will be $+\frac{x^4}{a^4}$, and the 6*th* term $-\frac{x^5}{a^5}$, &c., and so on, alternately *plus* and *minus*: this is called *the law of continuation* of the series. And the sum of all the terms when infinitely continued is said to be equal to the fraction $\frac{a}{a+x}$. Thus we say the vulgar fraction $\frac{2}{3}$ when reduced to a decimal, is $=\cdot6666$, &c. infinitely continued.

N. B. The terms in the quotient are found by dividing the remainders by (a) the first term of the divisor: thus, the first remainder $-x$ divided by a gives $-\frac{x}{a}$ the second term in the quotient; and the second remainder $+\frac{x^2}{a}$ divided by a gives $+\frac{x^2}{a^2}$ the third term, &c.

2. If the fraction is $\frac{a}{a-x}$, the series becomes wholly affirmative;

$$
\begin{array}{lllll}
\text{Thus } a-x\big)\, a & \Big(1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\text{\&c.} & & & \\
\phantom{\text{Thus } a-x\big)\,} \underline{a-x} & & & & \\
\phantom{\text{Thus } a-x\big)\,} +x & & & & \\
\phantom{\text{Thus } a-x\big)\,} \underline{+x-\frac{x^2}{a}} & & & & \\
\phantom{\text{Thus } a-x\big)\, +x} +\frac{x^2}{a} & & & & \\
\phantom{\text{Thus } a-x\big)\, +x} \underline{+\frac{x^2}{a}-\frac{x^3}{a^2}} & & & & \\
\phantom{\text{Thus } a-x\big)\, +x\,+\frac{x^2}{a}} +\frac{x^3}{a^2} & & & & \\
\phantom{\text{Thus } a-x\big)\, +x\,+\frac{x^2}{a}} \underline{+\frac{x^3}{a^2}-\frac{x^4}{a^3}} & & & & \\
\phantom{\text{Thus } a-x\big)\, +x\,+\frac{x^2}{a}\,+\frac{x^3}{a^2}} +\frac{x^4}{a^3} & & & &
\end{array}
$$

In this example, if $x < a$, the series is convergent, or the value of the terms continually diminish; but when $x > a$, it is said to diverge:

To explain this by numbers, let $a=3$, and $x=2$:

Then.......... $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}$, &c.

corresponding values $1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}$, &c. where the fractions or terms of the series grow less and less, and therefore the farther they are extended the more they converge or approximate to 0, which is supposed to be the last term or limit.

But if $a = 2$, and $x = 3$.

$$1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}, \ \&c.$$

corresponding values $1+\frac{3}{2}+\frac{9}{4}+\frac{27}{8}$, &c. in which the terms become larger and larger. This is called a diverging series.

If $x = 1$, and $a = 1$ in *Examp.* 1.

Then $\frac{a}{a+x} = 1-\frac{x}{a}+\frac{x^2}{a^2}=\frac{x^3}{a^3}$, &c.

will be $\frac{1}{1+1} = 1-1+1-1$, &c.

Now because $\frac{1}{1+1} = \frac{1}{2}$, it has been said that $1-1+1-1+1$, &c. *infinitely continued* is $= \frac{1}{2}$: —— a singular conclusion, when it is perceivable from the terms themselves, that their sum must necessarily be either 0, or $+1$, to whatever extent the division is supposed to be continued.

The real quotient however, results from the fractional part, which (by the division) is always $+\frac{1}{2}$ when the sum of the terms is 0; and $-\frac{1}{2}$ when the sum is $+1$: consequently $\frac{1}{2}$ is the true quotient in the former case; and $1-\frac{1}{2}$ in the other.

3. Let the fraction $\frac{a}{c-x}$ be expanded into a series.

$$\begin{array}{l}
c-x\big)\,a\quad\Big(\qquad \frac{a}{c}+\frac{ax}{c^2}+\frac{ax^2}{c^3}+\frac{ax^3}{c^4}+,\ \&c.\ \text{quotient.}\\
\qquad\ \underline{a-\frac{ax}{c}}\qquad\quad \text{or}\quad \frac{a}{c}\left(1+\frac{x}{c}+\frac{x^2}{c^2}+\frac{x^3}{c^3}+,\ \&c.\right)\\
\qquad\qquad +\frac{ax}{c}\\
\qquad\qquad \underline{+\frac{ax}{c}-\frac{ax^2}{c^2}}\\
\qquad\qquad\qquad +\frac{ax^2}{c^2}\\
\qquad\qquad\qquad \underline{+\frac{ax^2}{c^2}-\frac{ax^3}{c^3}}\\
\qquad\qquad\qquad\qquad +\frac{ax^3}{c^3}\\
\qquad\qquad\qquad\qquad \underline{+\frac{ax^3}{c^3}-\frac{ax^4}{c^4}}\\
\qquad\qquad\qquad\qquad\qquad +\frac{ax^4}{c^4}
\end{array}$$

By substituting other quantities for a, c, and x, in the quotient $\frac{a}{c}+\frac{ax}{c^2}+\frac{ax^2}{c^3}$, &c., different series may be produced.

Let $a=3$, $c=10$, and $x=1$:

then $\frac{a}{c-x}=\frac{a}{c}+\frac{ax}{c^2}+\frac{ax^2}{c^3}+\frac{ax^3}{c^4}+$ &c.

will be $\frac{3}{10-1}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+$ &c. which is the same series as the decimal answering to $\frac{1}{3}$ or $\frac{3}{10-1}$:

for $\frac{1}{3}=.3333$, &c. $=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$ &c.

4. To reduce $\frac{a^2+x^2}{a^4+x^4}$ to a series.

$$\begin{array}{l}
a^4+x^4)\ a^2+x^2\ \left(\frac{1}{a^2}-\frac{x^4}{a^6}+\frac{x^8}{a^{10}}-\&c.\right. \\
\qquad\quad \underline{a^2+\frac{x^4}{a^2}} \\
\qquad\qquad -\frac{x^4}{a^2}+x^2 \\
\qquad\qquad \underline{-\frac{x^4}{a^2}-\frac{x^8}{a^6}} \\
\qquad\qquad\qquad +\frac{x^8}{a^6}+x^2 \\
\qquad\qquad\qquad \underline{+\frac{x^8}{a^6}+\frac{x^{12}}{a^{10}}} \\
\qquad\qquad\qquad\quad -\frac{x^{12}}{a^{10}}+x^2
\end{array}$$

and the 4th term will be $-\frac{x^{12}}{a^{14}}$, the 5th $+\frac{x^{16}}{a^{18}}$, and so on.

73. It may be worth observing, that the same fraction will give different series if the order of the terms in its denominator is changed.

Thus taking Examp. 1.

$$\frac{a}{a+x}=1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\&c.$$

But $\frac{a}{x+a}=\frac{a}{x}-\frac{a^2}{x^2}+\frac{a^3}{x^3}-$ &c.

The two quotients however, will always be equal when the remainders are taken into the account.

OF EQUATIONS.

74. When the symbol = is placed between two quantities that are equal, but differently expressed, it is called an Equation. (3.)

Thus, $7+2$ and $10-1$ are equal;

And $7+2=10-1$ is an equation denoting the equality of $7+2$ and $10-1$.

Also $x=c-d$ is an equation which shews that the quantity x is equal to the difference of the quantities c and d.

Equations take their denominations from the highest power of the unknown quantity which they involve: For example, if that quantity is of one dimension only, it is called a Simple Equation; if of two dimensions, a Quadratic; when of three, a Cubic, &c.

Thus, if x be the unknown quantity;
then $10x-7=4x+a$ is a simple equation.
$ax^2-2x=cd$ a quadratic equation.
$x^3+x^2-cx=a^2b$ a cubic equation.
$x^4+ax^3+dx=abc$ a biquadratic.
$x^m-x^{m-1}+cx^{m-2}=ab$ an equation of m dimensions.

75. When x, y, z, &c. or other symbols, are put to denote unknown quantities, it is from certain given relations they have to each other, and to such as are known, that equations are derived. And to resolve or reduce an equation, is to discover the value of the unknown quantity which it involves: but no rule has yet been found sufficiently general for that purpose in all cases. The resolution of Simple and Quadratic Equations however, principally depend on the following obvious

AXIOMS:

1. If equal quantities are added to equal quantities, the sums are equal.

2. If equal quantities are subtracted from equal quantities, the remainders are equal.

3. If equal quantities are multiplied by equal quantities, the products are equal.

4. If equal quantities are divided by equal quantities, the quotients are equal.

5. If two or more quantities are each equal to another quantity, those quantities are equal.

6. Like powers of equal quantities are equal.

7. Like roots of equal quantities are equal.

8. A whole quantity is equal to all its parts taken together.

76. RESOLUTION OF SIMPLE EQUATIONS.

Examples.

1. GIVEN $x-7=22$; to find the value of x.

By adding 7 to each side of the equation

we have $x-7+7=22+7$ (*Ax.* 1.)
or $x=29$ the value required.

Therefore, any quantity may be transposed from one side of an equation to the other, by changing its sign.

For $x-7=22$:

And $x=22+7$, where 7 is transposed from one side of the equation to the other, and its sign changed from $-$ to $+$.

In like manner, if $x-a=b$
then $x=b+a$.

77. Hence also it appears, that if all the signs in an equation are changed, the equality still subsists:

Thus, changing the signs of $x = b + a$
And it becomes............ $-x = -b - a$
Now by transposing a... $-x + a = -b$
Next, transpose x........ $a = -b + x$
Lastly, transpose b.... then $a + b = x$, as before.

2. If $x + 8 = 18$; what is the value of x?

Let 8 be subtracted from each side of the equation;

Then $x + 8 - 8 = 18 - 8$ (*Ax.* 2.)
And $x = 10$ the answer.

Or thus. Since $x + 8 = 18$; then $x = 18 - 8 = 10$, the answer.

3. Given $\frac{x+7}{3} = 13$; to find x.

If each side of the equation be multiplied by 3,

we have $\frac{(x+7) \times 3}{3} = 39$ (*Ax.* 3.)
or $x + 7 = 39$ (56)
And $x = 39 - 7 = 32$ the answer.

In like manner, if $\frac{x+a}{c} = b$,

then $\frac{(x+a) \times c}{c} = bc$ (*Ax.* 3.)
or $x + a = bc$ (56)
And $x = bc - a$.

4. Suppose $3x + 14 = 50$; what is the value of x?

$3x + 14 = 50$
then ... $3x = 50 - 14$
or ... $3x = 36$
And dividing by 3 gives ... $x = \frac{36}{3}$ (*Ax.* 4.)
or .... $x = 12$ the value required.

And if......... $ax + b = c$
then ... $ax = c - b$
And dividing by the coefficient a... $x = \frac{c-b}{a}$ the value of x.

5. If $\frac{7x - 45}{4} = x$; what is the value of x?

By multiplying each side of the equation by 4

we get.... $\frac{(7x - 45) \times 4}{4} = 4x$

or.......... $7x - 45 = 4x$

by transposing 45...... $7x = 4x + 45$

Subtracting $4x$ from each side ...... gives $3x = 45$

And dividing by the coefficient 3 (*Ax.* 4.)... $x = 15$ the answer.

6. Let $\frac{x + 5}{2} - 2x = \frac{x}{3}$: required x?

In order to clear the equation from fractions, let both sides be first multiplied by 3,

and we have.......................... $\frac{3x + 15}{2} - 6x = x$

and multiplying again by 2....gives $3x + 15 - 12x = 2x$

But $3x - 12x = -9x$, therefore we have $15 - 9x = 2x$

And transposing $9x$......gives $15 = 11x$

And $x = \frac{15}{11} = 1\frac{4}{11}$ — the answer.

Or thus:

The given equation is $\frac{x + 5}{2} - 2x = \frac{x}{3}$

Which is the same as ...... $\frac{1}{2}x + 2\frac{1}{2} - 2x = \frac{1}{3}x$

Now transposing $2\frac{1}{2}$ .... gives $\frac{1}{2}x - 2x = \frac{1}{3}x - 2\frac{1}{2}$

But $\frac{1}{2}x - 2x = -1\frac{1}{2}x$ ... therefore $-1\frac{1}{2}x = \frac{1}{3}x - 2\frac{1}{2}$

And transposing $\frac{1}{3}x$ .... gives $-1\frac{1}{2}x - \frac{1}{3}x = -2\frac{1}{2}$

or $-1\frac{5}{6}x = -2\frac{1}{2}$

And dividing by the coefficient $-1\frac{5}{6}$ gives $x = 1\frac{4}{11}$ as before.

7. Given $\frac{5x + 7x - 9x}{4} = 100$; to find x

Multiplying both sides by 4 gives $5x + 7x - 9x = 400$

or $12x - 9x = 400$

or $3x = 400$

And $x = \frac{400}{3} = 133\frac{1}{3}$ Ans.

Or thus.

$$5x + 7x - 9x = (5 + 7 - 9)x$$

therefore $(5 + 7 - 9)\ x = 400$

then (*Ax.* 4.) $x = \frac{400}{5 + 7 - 9}$.

And according to this last method the value of the unknown quantity is denoted when it has literal coefficients:

For let $\frac{ax + bx - cx}{4} = m$

Then $ax + bx - cx = 4m$

or $(a + b - c)\ x = 4m$

Therefore $x = \frac{4m}{a + b - c}$. (*Ax.* 4).

8. Given $\frac{a}{x} - \frac{b}{x} + \frac{c}{x} = d$; to find x.

Both sides of the equation multiplied by x gives $a - b + c = dx$

And dividing both sides by d ...... $\frac{a - b + c}{d} = x$.

9. Given $\frac{3x}{17 - 4x} = 19$; to find x.

Both sides multiplied by $17 - 4x$ gives $3x = 19\ (17 - 4x)$

or $3x = 323 - 76x$

And transposing $76x$...... $3x + 76x = 323$

or........ $79x = 323$

And........ $x = \frac{323}{79} = 4\frac{7}{79}$ Ans.

10. Given $\frac{ax + bx}{x - c} = d - n$; to find x.

Let both sides of the equation be multiplied by $x - c$;

Then............ $ax + bx = (d - n)\ (x - c)$

or $ax + bx = cn - dc + dx - nx$

And transposing dx and nx... $ax + bx + nx - dx = cn - dc$

or $(a + b + n - d)\ x = cn - dc$

Therefore.............. $x = \frac{cn - dc}{a + b + n - d}$. Ans.

11. Let $a^2 - x^2 = bx + ba$; to find x.

When each side of the Equation is resolved into its factors, we have......... $(a+x)\ (a-x)=b\ (x+a)$

Then dividing by $a+x$ gives $a-x=b$

And transposing x and b, the result is... $a-b=x$.

12. Given $\frac{ax^2+ac^2}{a+x}=ax+b^2$; to find x.

Multiplying by $a+x$ gives... $ax^2+ac^2=(ax+b^2)\ (a+x)$

or $ax^2+ac^2=a^2x+ax^2+b^2x+ab^2$

And subtracting ax^2 from each side,...... $ac^2=a^2x+b^2x+ab^2$

And transposing ab^2.............. $ac^2-ab^2=a^2x+b^2x$

Then dividing by the coefficient a^2+b^2 gives $\frac{ac^2-ab^2}{a^2+b^2}=x$.

13. Let $a-x=\frac{x^2}{a-x}$; to find x.

If each side be multiplied by $a-x$, we have $a^2-2ax+x^2=x^2$

And subtracting x^2 from each side gives $a^2-2ax=0$

by transposing $2ax$.......... $a^2=2ax$

And dividing by $2a$ gives $\frac{a^2}{2a}=\frac{a}{2}=x$.

Of reducing Simple Equations when the values of two unknown quantities are required.

78. If two independent equations are given, which involve two unknown quantities, find two expressions for one of them, one from each equation, by the foregoing methods; those expressions being put equal, an equation will arise with only one unknown quantity in it, whose value may be found as before.

Examples.

1. Given $3x-4y=1$
$7x+3y=64$. To find x and y.

First,............. $3x-4y=1$

By transposition $3x=1+4y$

therefore $x=\frac{1+4y}{3}$.

Secondly, ... $7x+3y=64$

By transposition...... $7x=64-3y$

and... $x=\frac{64-3y}{7}$

Therefore $\frac{1+4y}{3}=\frac{64-3y}{7}$ (*Ax.* 5).

This equation cleared of fractions (Examp. 6.) gives $7+28y=192-9y$.

And by transposition $37y=185$

or $y=\frac{185}{37}=5$.

And substituting 5 for y gives $x=\frac{1+4y}{3}=\frac{1+20}{3}=7$ the value of x.

79. But it will frequently be more expeditious to multiply, or divide the equations by such numbers or quantities as will make the term which contains one of the unknown quantities the same in both equations; then by adding, or subtracting the equations, as the case may require, that term will be exterminated.

Thus, if the first equation in the preceding example be multiplied by 7, and the second by 3,

we have.... .. .. $21x-28y=7$

and .... $21x+9y=192$

The upper subtracted from the lower gives $37y=185$ (*Ax.* 2.).

Therefore ...... $y=\frac{185}{37}=5$, as before.

Again, if we would exterminate y, multiply the first equation by 3, and the second by 4:

Then................ $9x-12y=3$

and $28x+12y=256$

The sum is...... $37x \quad =259$ (*Ax.* 1.)

whence $x=\frac{259}{37}=7$, as before.

2. Let $az+bx=c$

$dz+gx=p$. To find z and x.

The first equation multiplied by d, and the second by a.

gives $dax + dbx = dc$

and $dax + agx = ap$

And subtracting the lower from the upper $dbx - agx = dc - ap$ (*Ax.* 2.)

And dividing by the coefficient $db - ag$.... ... $x = \frac{dc - ap}{db - ag}$.

But to exterminate x, let the first equation be multiplied by g, and the second by b;

Then ... . $gax + gbx = gc$

and .... $bdz + gbx = bp$

Subtracting the under from the upper...... $gaz - bdz = gc - bp$

whence......... $z = \frac{gc - bp}{ga - bd}$

Remark. In this example there is nothing to indicate which of the two equations is greatest, and consequently we are at liberty to subtract the upper from the lower; in that case $x = \frac{ap - dc}{ag - db}$; and $z = \frac{bp - gc}{bd - ga}$. The same expressions however, result from the equations by changing their signs:

For if $dbx - agx = dc - ap$, then (77) $agx - dbx = ap - dc$, whence $x = \frac{ap - dc}{ag - db}$,

and $gaz - bdz = gc - bp$, then $bdz - gaz = bp - gc$, and $z = \frac{bp - gc}{bd - ga}$.

But such expressions are usually set down thus $x = \frac{ap \backsim dc}{ag \backsim db}$, and $z = \frac{bp \backsim gc}{bd \backsim ga}$.

3. Given $\frac{7x}{2} + z = 52$

$z - \frac{7x}{16} = 16$. To find x and z.

From the second equation we have $z = 16 + \frac{7x}{16}$

which substituted for z in the first, gives... $\frac{7x}{2} + 16 + \frac{7x}{16} = 52$

or $\frac{7x}{2} + \frac{7x}{16} = 52 - 16 = 36$

Whence $56x + 7x = 36 \times 16 = 576$

or $63x = 576$

and $x = 9\frac{1}{7}$

Now put $9\frac{1}{7}$ for x, and we have $z = 16 + \frac{7x}{16} = 16 + \frac{7 \times 9\frac{1}{7}}{16} = 20$ the value of z.

4. Given $x^2 - z^2 = 3876$
$x + z = 102$. To find x and z.

Divide the first equation by the second... $\frac{x^2 - z^2}{x+z} = \frac{3876}{102}$ (*Ax.* 4.)
which, by actual division............ gives $x - z = 38$
whence $x = 38 + z$
And putting $38 + z$ for x in the second equation... $38 + z + z = 102$
or $2z = 102 - 38$
Therefore $z = 32$
And $x = 38 + z = 38 + 32 = 70$

When Three Equations are given, involving Three unknown quantities.

80. If the three unknown quantities are found in all the equations, find three expressions for one of them, one from each equation, then compare the first expression with the second, and also with the third, by which means that quantity will be exterminated, and the equations reduced to two; which may be resolved as in the preceding articles. But the method in Art. 79, will generally be found the least tedious.

Examples.

1. Given $x + y = 19$
$x + z = 20$
$z + y = 21$. To find x, y, and z.

Subtracting the first equation from the second gives $z - y = 1$
To this remainder add the third equation, and we have $2z = 22$
whence $z = 11$.

Now putting 11 for z in the 2*d*. and 3*d*. equations,
we have $11 + y = 21$, and $x + 11 = 20$; whence $y = 10$; and x [illegible] 9.

2. Let $x + y + z = 10$
$x + 2y + 3z = 23$
$2x + 3y + 5z = 38$. To find x, y, and z.

By the first equation $x = 10 - y - z$
From the second........... $x = 23 - 2y - 3z$
therefore $10 - y - z = 23 - 2y - 3z$
whence $y = 13 - 2z$.

Next, by the 3*d.* equation $x = \frac{38 - 3y - 5z}{2}$
And from the first ...... $x = 10 - y - z$
therefore... $10 - y - z = \frac{38 - 3y - 5z}{2}$
whence $20 - 2y - 2z = 38 - 3y - 5z$
and......... $y = 18 - 3z$
But......... $y = 13 - 2z$
therefore $18 - 3z = 13 - 2z$
and.... $z = 5$.

Now substituting 5 for z in the two first equations, we have $x + y = 5$, and $x + 2y = 8$; whence $x = 2$, and $y = 3$.

Or thus:

Subtracting the 1*st.* equation from the 2*d.* gives $y + 2z = 13$
and double the 2*d.* from the 3*d.* gives $-y - z = 38 - 46 = -8$
or $y + z = 8$ (77)
whence. .. $y = 8 - z$
But... $y + 2z = 13$
and... $y = 13 - 2z$
Therefore $8 - z = 13 - 2z$
which gives $z = 5$, as before.

3. Given $x + \frac{y}{2} + \frac{z}{3} = 32$
$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 15$
$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 12$. To find x, y, and z.

The equations cleared of fractions become

$$6x + 3y + 2z = 192$$
$$20x + 15y + 12z = 900$$
$$15x + 12y + 10z = 720.$$

To exterminate y (for example) let the second equation be subtracted from 5 times the first, and the third equation from 4 times the first:

$$\begin{array}{r l}
\text{5 times the first} & 30x + 15y + 10z = 960 \\
\text{the second} & 20x + 15y + 12z = 900 \\ \hline
\text{remainder} & 10x \quad - 2z = 60 \\ \hline
\end{array}$$

$$\begin{array}{r l}
\text{4 times the first} & 24x + 12y + 8z = 768 \\
\text{the third} & 5x + 12y + 10z = 720 \\ \hline
\text{remainder} & 9x \quad - 2z = 48
\end{array}$$

Then the last remainder subtracted from the first gives $x = 60 - 48 = 12$. Now substituting 12 for x in the first remainder, we have $10 \times 12 - 2z = 60$, whence $z = 30$; then from the first equation y is found $= 20$.

4. Given $ax + by + cz = m$

$$dx + gy + hz = n$$

$fx + ry + sz = p$. To find x, y, and z in terms or functions of the other quantities.

If we begin with exterminating z, let the first equation be multiplied by the product hs (the coefficients of z in the other equations), the second equation by cs, and the third by hc;

and we have $hsax + hsby + hscz = hsm$

$$csdx + csgy + cshz = csn$$

$hcfx + hcry + hcsz = hcp$: where the coefficients of z are the same.

Now subtracting the first of these equations from the second, and also from the third, the results will be

$$csdx - hsax + csgy - hsby = csn - hsm$$

$$hcfx - hsax + hcry - hsby = hcp - hsm$$

or $cdx - hax + cgy - hby = cn - hm$

and $cfx - sax + cry - sby = cp - sm$ (by dividing the first by s, and the latter by h).

From the first of these equations... $x = \frac{cn - hn - cgy + hby}{cd - ha}$

And from the second...... $x = \frac{cp - sm - cry + sby}{cf - sa}$

Therefore $\frac{cn - hm - cgy + hby}{cd - ha} = \frac{cp - sm - cry + sby}{cf - sa.}$:

Which reduced gives $y = \frac{fhm - dsm + san - cfn + cdp - hap}{fhb - dsb + sag - cfg + cdr - har}$, (A)

or $y = \frac{(fh - ds)m + (sa - cf)n + (cd - ha)p}{(fh - ds)b + (sa - cf)g + (cd - ha)r}$. (B)

From this value of y, the expressions for x and z may be obtained without substitution, or repeating the process :

For it is evident from *Ex.* 2. (Art. 79.) that the expressions for x and z will have the same denominator as this for y. And since the coefficients of m, n, and p, in the expression (B) are respectively the same, and have the same signs as those of b, g, and r in the denominator (these latter being the coefficients of y in the given equations), it is manifest from analogy that m, n, and p will have the same respective coefficients and signs in the required numerators as a, d, and f; and c, h, and s have in the denominator: a, d, and f; and c, h, and s being the coefficients of x and z in the given equations, following the same order as those of y.

Now the given denominator (A) when resolved into factors exhibiting those coefficients

will be $(gs - rh)\,a + (rc - bs)\,d + (bh - cg)\,f$, for x:

and $(dr - gf)\,c + (fb - ra)\,h + (ag - bd)\,s$, for z.

$$\text{Therefore } x = \frac{(gs - rh)\,m + (rc - bs)\,n + (bh - cg)\,p}{(gs - rh)\,a + (rc - bs)\,d + (bh - cg)\,f}.$$

$$\text{And } \quad z = \frac{(dr - gf)\,m + (fb - ra)\,n + (ag - bd)\,p}{(dr - gf)\,c + (fb - ra)\,h + (ag - bd)\,s}.$$

Instead of subtracting the first equation from the other two, a contrary order might have been adopted; for that reason, perhaps, the symbol $\backsim$ would be more proper than the negative sign in the final expressions: See Examp. 2. (79.)

The last example is sufficient to direct the process, when four or more unknown quantities are concerned. But methods of reduction different from those we have given, will frequently present themselves in practice.

81. When the number of equations is less than the number of unknown quantities they involve, the problem is said to be indeterminate or unlimited. Thus if $x + y = 10$, then x and y may be any two numbers whose sum is 10. Or suppose $x - y = 6$, and $z - x = 9$, in which case y may be any number whatever; and consequently the values of the three unknown quantities will be indefinite. The like must also take place

when the number of equations and unknown quantities are the same, if one of the equations is deducible from the others:

Thus in *Ex.* 4. let $a=1$, $b=2$, $c=3$, $d=4$, $g=5$, $h=6$, $f=7$, $r=8$, $s=9$, $m=20$, $n=47$, and $p=74$:

Then the three equations become
$$x+2y+3z=20$$
$$4x+5y+6z=47$$
$$7x+8y+9z=74.$$

Now substituting those numbers in the expressions for the values of x, y, and z, the numerators and denominators become $=0$, or the expressions vanish. The reason perhaps is not obvious at first sight; but on examining the equations we find, that double the second is equal to the sum of the other two, and consequently there are only *two* independent equations. Also, with these numeral coefficients, $2n$ must be $=m+p$, otherwise the equations are incongruous.

82. Sometimes equations may involve an absurdity; as when $x-y=8$, and $x+y=7$; for it is impossible that the difference of two quantities should be greater than their sum.

The young Algebraist will now perceive, that the art of resolving Equations consists in bringing each of the unknown quantities on one side of an equation having known quantities on the other.

OF RATIOS AND PROPORTIONS.

83. THE relation or proportion which two quantities of the same kind bear to each other in respect of magnitude, is called the Ratio of those quantities: this is found by considering what part or parts one is of the other, or how often one is contained in the other.

Thus if $12a$ and $4a$ are the two quantities, then by comparing them, we find their magnitudes such, that the former contains the latter 3 times, and in common language we say it is 3 times as big, because 4 is contained 3 times in 12: the quantities therefore appear to have the same ratio or proportion, the greater to the less, as 3 has to 1. Hence it is, that the equality of two Ratios constitutes Proportion.

The terms of the two equal ratios are sometimes set down thus:

$12a : 4a = 3 : 1$; *viz.* the ratio of $12a$ to $4a$ is equal to that of 3 to 1.

Or thus, $12a : 4a :: 3 : 1$, which may be read thus — $12a$ bears the same proportion to $4a$ as 3 does to 1; or, As $12a$ is to $4a$, so is 3 to 1.

The 4th term 1 is called a *4th* proportional to the other three.

The Antecedents of the two ratios are $12a$ and 3, and their consequents $4a$ and 1.

84. The terms of the ratio $3 : 1$ are like submultiples of $12a : 4a$, the divisor being $4a$. But any other like submultiples or multiples of $12a$ and $4a$ will have the same ratio or proportion; for $\frac{12a}{4a} = \frac{3}{1} = \frac{6a}{2a} = \frac{3a}{a} = \frac{24a}{8a} = \frac{15ab}{5ab}$, &c. where each numerator has the same ratio to its denominator as $12a$ has to its denominator $4a$. This is evident from the nature of fractions.

Hence $12a : 4a :: 3 : 1 :: 6a : 2a :: 3a : a :: 24a : 8a :: 15ab ; 5ab$, &c., are a rank of proportionals.

85. The fraction $\frac{12a}{4a}$ or the antecedent divided by the consequent, is by many authors, called the magnitude or quantity of

ratio of $12a$ to $4a$; and sometimes its measure, or exponent. But writers differ on this subject. Some will have the ratios (when considered as magnitudes,) to be the exponents of the powers of their terms. Thus if 1 denotes the magnitude of the ratio of a to b, that of the ratio of a^2 to b^2 will be 2; that of a^3 to b^3 will be 3, &c. these indices or ratios are therefore analogous to the scale of Logarithms; and consequently the ratio of equality will be 0; for if $a = b$, then $\frac{a}{b} = 1$, whose logarithm is 0. Dr. Barrow however, says, "Reason" (Ratio) "is not quantity;" and maintains that the magnitude of a simple Ratio cannot be expressed in numbers: but if its quantity be referred to the fraction formed by the two terms, then he makes the magnitude of the ratio of equality greater than 0. See his *Math. Lectures.*

If the terms of the ratio however, are commensurable, the ratio itself may be expounded by the quotient arising from the division of the antecedent by the consequent, as in Arithmetic. Thus in the progression 128, 64, 32, 16, 8, &c. we call 2 the common ratio of the terms, $\left(\text{for } \frac{128}{64} = 2, \text{ or } \frac{64}{32} = 2, \text{ \&c}\right)$. Also in the progression ar^3, ar^2, ar, a, $\frac{a}{r}$, $\frac{a}{r^2}$, &c. r is the common ratio. But when the terms are incommensurable, the ratio cannot be exhibited in this manner: Thus it is impossible to find in numbers the exact ratio, or proportion which the square root of 1 has to the square root of 2. And in comparing geometrical magnitudes, the ratios are not set down fractionwise, except the terms are supposed to be subjected to some common measure.

86. When the terms of two ratios are commensurable, the greatest of the two may be found thus: Let $7a$ and $8a$ be the terms of one ratio, and $8b$ and $9b$ the terms of the other, in the same order;

Then expounding the ratios by the fractions $\frac{7a}{8a}$ and $\frac{8b}{9b}$, and reducing them to a common denominator, we get $\frac{7a}{8a}=\frac{63ab}{72ab}$, and $\frac{8b}{9b}=\frac{64ab}{72ab}$ which is greater than $\frac{63ab}{72ab}$, therefore the ratio $8b$ to $9b$ is greater than that of $7a$ to $8a$, or the ratio 7 to 8 less than that of 8 to 9.

87. Ratios equal to the same, or to equal ratios, are equal to each other.

Thus if $a:b::c:d$
and $g:h::c:d$
Then $a:b::g:h$ (75. *Ax.* 5.)

Or thus, since $\frac{a}{b}=\frac{c}{d}$

and $\frac{g}{h}=\frac{c}{d}$, therefore $\frac{a}{b}=\frac{g}{h}$.

88. If 4 quantities are proportional, the product of the means is equal to the product of the extremes.

Thus suppose $a:b::c:d$
Then $ad=bc$.

For $\frac{a}{b}=\frac{c}{d}$ (84), and multiplying both fractions by bd

we have $\frac{abd}{b}=\frac{cbd}{d}$ (75. *Ax.* 3), or $ad=bc$, by reducing the fractions.

Cor. 1. Hence the terms may be any how varied so that a and d, or b and c are the extremes (135 Arith.)

Thus $b:a::d:c$
$d:c::b:a$, &c. &c. for in either case $ad=bc$.

Cor. 2. When two fractions $\left(\frac{a}{b}=\frac{c}{d}\right)$ are equal, their reciprocals are equal $\left(\frac{b}{a}=\frac{d}{c}\right)$.

Cor. 3. If the product of two factors is equal to the product of two other factors, the 4 factors are proportional.

Thus suppose $(a+b)x=(c+d)z$,
Then $a+b:c+d::z:x$.

Cor. 4. Hence a proportion may be converted into an Equation:

Thus, let $a+x:c::ab:x$.
Then $ax+x^2=cab$.

N. B. If $a:b::c:d$; then $b:a::d:c$ is called *inversely*; and $a:c::b:d$ *alternately*.

89. When 4 quantities are proportional; Then, as the sum of the first and second, is to the first (or second), so is the sum of the third and fourth, to the third (or fourth),

Let $a:b::c:d$
Then $a+b:a::c+d:c$.
And $a+b:b::c+d:d$.

Because $\frac{b}{a}=\frac{d}{c}$ (88. *Cor.* 2.), if we add $\frac{a}{a}$ to the first fraction, and $\frac{c}{c}$ to the second (or 1 to each)

we have $\frac{b}{a}+\frac{a}{a}=\frac{d}{c}+\frac{c}{c}$ (75. *Ax.* 1.)

or $\frac{b+a}{a}=\frac{d+c}{c}$, and since the terms of two equal fractions are proportional,

$b+a:a::d+c:c$. This is said to be by *composition*.

In like manner, since $\frac{a}{b}=\frac{c}{d}$, if $\frac{b}{b}$ and $\frac{d}{d}$ are added to the two fractions, respectively,

we get $\frac{a+b}{b}=\frac{c+d}{d}$,
therefore $a+b:b::c+d:d$.

90. And by subtracting $\frac{a}{a}$, $\frac{c}{c}$, &c. (instead of the addition) we shall have $\frac{b-a}{a}=\frac{d-c}{c}$,

whence $b-a:a::d-c:c$

Therefore, as the difference of the two first terms, is to the first (or second), so is the difference of the third and fourth terms, to the third (or fourth). This is called *division*.

Corol. Because $b + a : d + c :: a : c$
and $b - a : d - c :: a : c$,

Therefore by equality, $b + a : b - a :: d + c : d - c$. (87.)

91. If there be any number of proportional quantities, Then either antecedent, is to its consequent, as the sum of all the antecedents, to the sum of all the consequents.

Let $a : b :: c : d :: f : g :$
Then $a : b :: c : d$, hence $ad = bc$
$a : b :: f g$ .... $ag = bf$
Therefore $ad + ag = bc + bf$ (75. *Cor.* 1.)
or $a\,(d + g) = b\,(c + f)$
Whence $a : c + f :: b : d + g$ (88. *Cor.* 3.)
And $a + c + f : a :: b + d + g : b$ (89.)
Or alternately, $a + c + f : b + d + g :: a : b$ ($c : d$, &c.)

Cor. Because $\frac{a}{b} = \frac{c}{d} = \frac{f}{g}$, if there be any number of equal fractions, then, as either numerator, is to its denominator, so is the sum of any two or more of the numerators, to the sum of their corresponding denominators.

92. Let there be four proportional quantities; Then if like multiples or submultiples be taken of all the quantities, or of the first and second, or the third and fourth, or of the antecedents, or consequents; in either case the resulting terms will still be proportional.

This is manifest from Art. 84, or from *Ax*. 3 and 4, Art. 75.

Thus, let $a : b :: c : d$; then $\frac{a}{b} = \frac{c}{d}$.
And $na : nb :: nc : nd$.
$\frac{a}{n} : \frac{b}{n} :: \frac{c}{n} : \frac{d}{n}$.

$$na : nb :: c : d.$$
$$na : b :: nc : d.$$
$$\frac{a}{n} : b :: \frac{c}{n} : d.$$
$$na : \frac{b}{n} :: nc : \frac{d}{n}.$$
$$na : nb :: mc : md.$$
&c. &c.

For in each proportion, the antecedents divided by their consequents form equal fractions.

93. In four proportional quantities, if the two consequents be either augmented or diminished by quantities that have the same ratio as the respective antecedents, the results, and the antecedents will still be proportionals.

Let $a : b :: c : d$

Then $na : b :: nc : d$ (by the preceding Art.)

And $b \pm na : na :: d \pm nc : nc$ (89. 90).

Or alternately, $b \pm na : d \pm nc :: na : nc$.

Whence $b \pm na : d \pm nc :: a : c$, by taking equal submultiples of na and nc.

94. In a proportion, if the second and third terms are the same, the product of the first and fourth is equal to the square of the second.

Let $a : b :: b : c$

Then $ac = b^2$ (88).

The fourth term c is called a third proportional to a and b. And b is a mean proportional between a and c. Also the three terms are continued proportionals.

95. Like Powers, or Roots, of proportional quantities, are also respectively proportionals.

Suppose $a : b :: c : d$,

Then $\frac{a}{b} = \frac{c}{d}$

And $\frac{a^2}{b^2}=\frac{c^2}{d^2}$ (75. *Ax.* 3, or 5.)

Whence $a^2 : b^2 :: c^2 : d^2$.

And if $p^3 : g^3 :: r : s$;

Then $\frac{p^3}{g^3}=\frac{r}{s}$; and $\frac{p}{g}=\frac{r^{\frac{1}{3}}}{s^{\frac{1}{3}}}$, by taking the cube roots of the fractions.

Whence $p : g :: r^{\frac{1}{3}} : s^{\frac{1}{3}}$.

Generally, if $a : b :: c : d$; then $a^m : b^m :: c^m : d^m$, where m may be any number, whole or fractional.

The ratio of two squares is called *duplicate ratio*; of two square roots, *subduplicate*; of two cubes, *triplicate ratio*; and of two cube roots, *subtriplicate*, &c. Thus if a and A denotes the areas of two circles, and d and D their respective diameters:

Then $d^2 : D^2 :: a : A$ (Geom. 105. *Cor.*) that is, the areas are in the duplicate ratio of their diameters.

And taking the square roots of the four terms,

$d : D :: a^{\frac{1}{2}} : A^{\frac{1}{2}}$, *viz.* the diameters are in the subduplicate ratio of the areas.

Also, if a and A are the solid contents of two spheres, and d and D their diameters,

Then $d^3 : D^3 :: a : A$ (Geom. 135. *Cor.* 3.)

And $d : D :: a^{\frac{1}{3}} : A^{\frac{1}{3}}$,

Or, The solid contents of spheres are in the triplicate ratio of their diameters:

And, The diameters of spheres are in the subtriplicate ratio of their solid contents.

96. If there be several ranks of proportional quantities, then the products of the corresponding terms will be proportionals.

Let $a : b :: c : d$;
And $f : g :: h : k$
Then $af : bg :: ch : dk$.

For $\frac{a}{b}=\frac{c}{d}$, and $\frac{f}{g}=\frac{h}{k}$:

And $\frac{a}{b}\times\frac{f}{g}=\frac{c}{d}\times\frac{h}{k}$, or $\frac{af}{bg}=\frac{ch}{dk}$ (75. *Ax.* 3.)

Whence $af:bg::ch:dk$. And so for any number of ranks.

This is called *compounding* the proportions.

97. In a rank of continued proportionals, the ratio of the first term to the last is compounded of the ratios of all the antecedents to their consequents.

Let $a:b::b:c::c:d::d:f$

Then $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{f}$, and the continued product

is $\frac{abcd}{bcdf}$, which in its lowest terms is $\frac{a}{f}$, the ratio of a to f.

Cor. Hence the ratio of the first term to the third will be equal to the ratio of the squares of the first and second.

For $b^2=ca$
And $ab^2=ca^2$
Whence $a:c::a^2:b^2$:

And the ratio of the first to the fourth is equal to the ratio of the cubes of the first and second.

Because $a:c::a^2:b^2$,
Therefore $\frac{a}{c}=\frac{a^2}{b^2}$: But $\frac{c}{d}=\frac{a}{b}$:
Whence $\frac{a}{c}\times\frac{c}{d}=\frac{a^2}{b^2}\times\frac{a}{b}$, or $\frac{a}{d}=\frac{a^3}{b^3}$; (75. *Ax.* 3.)

Therefore $a:d::a^3:b^3$. And in like manner it is proved that $a:f::a^4:b^4$. And so of others.

And the first term is said to have to the third, the duplicate ratio of that which it has to the second; and to the fourth, the triplicate ratio of that which it has to the second; and so on.

N. B. This compounding of ratios by multiplication is called addition of ratios by those who consider ratios to be the exponents of the powers of their terms.

98. If there be four proportional quantities of the same kind, the sum of the least and greatest is greater than the sum of the other two.

Suppose $a : b :: c : d$; and let a be the least, and d the greatest. Now if the quantities are commensurable, c and d will be like multiples, or submultiples, of a and b, respectively:

Therefore if c and d are expounded by $(n+1)a$, and $(n+1)b$,

we have $a : b :: na+a : nb+b$.

And $nb+b+a$ is the sum of the least and greatest terms $(a+d)$.

$na+a+b$ is the sum of the other two $(b+c)$; which is less than the other sum, because b is greater than a, by the hypothesis, and therefore nb greater than na.

OF INVOLUTION.

99. If a quantity be continually multiplied by itself, it is said to be involved or raised to a power equal to the number of times it has been employed in the multiplication.

Thus $a \times a = a^2$, the 2*d*. power, or square.

$a \times a \times a = a^3$, the 3*d*. power, or cube.

$a \times a \times a \times a = a^4$, the 4*th*. or biquadrate.

$a^n =$ the *n*th. power. (42.)

Here $+a$ is the root; and all the powers are positive. But if the root is negative, then its odd powers, or the 3*d*, 5*th*, 7*th*, &c. will be negative.

For $-a \times -a = a^2$, the square (44).

$a^2 \times -a = -a^3$, the cube of $-a$.

$-a^3 \times -a = a^4$, the 4*th*. power.

$a^4 \times -a = -a^5$, the 5*th*. &c.

100. But simple quantities are raised to any power by multiplying the index of every factor in the quantity by the exponent of the power.

Thus the square of a^2 is $a^{2\cdot2} = a^4$

The cube of a^2 is $a^{2\cdot3} = a^6$.

The cube of ab^2 is $a^{1\cdot3}\, b^{2\cdot3} = a^3b^6$. For $ab^2 \times ab^2 \times ab^2 = a^3b^6$.

Also $-a^m$ raised to the *n*th. power is $+a^{mn}$ or $-a^{mn}$, according as *n* is even, or odd.

And $3a^{-4}$ raised to the *n*th. power is $3^n a^{-4n}$.

101. Fractions are raised to given powers by involving their terms:

Thus the square of $\frac{3ab}{4cd}$ is $\frac{9a^2b^2}{16c^2d^2}$.

And the cube of $\frac{2ab^2}{xy}$ is $\frac{8a^3b^6}{x^3y^3}$.

102. Compound quantities are involved by actual multiplication, as in Art. 44:

Thus if the root be $a+b$:

$$
\begin{array}{rl}
 & a+b \\
 & \underline{a+b} \\
 & a^2+ab \\
 & \underline{\quad +ab+b^2} \\
(a+b)^2 = & a^2+2ab+b^2 \text{ the square or 2d. power.} \\
 & \underline{\qquad\qquad a+b} \\
 & a^3+2a^2b+ab^2 \\
 & \underline{\quad +a^2b+2ab^2+b^3} \\
(a+b)^3 = & a^3+3a^2b+3ab^2+b^3 \text{ the cube or 3d. power.} \\
 & \underline{\qquad\qquad\qquad a+b} \\
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \underline{\quad +a^3b+3a^2b^2+3ab^3+b^4} \\
(a+b)^4 = & a^4+4a^3b+6a^2b^2+4ab^3+b^4 \text{ the 4th. power.} \\
 & \qquad \&c.
\end{array}
$$

If the root be $a-b$ then the terms which involve the odd powers of b will be negative, viz. the signs are alternately *plus* and *minus*:

Thus $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$, the exponent of b being an odd number in the 2*d*. and 4*th*. terms.

Sometimes it is most convenient to represent the powers by the indices; Thus $(ax - x)^3$ denotes the third power of $ax - x$.

OF EVOLUTION OR THE EXTRACTION OF ROOTS.

103. THIS is the reverse of Involution, and consists in discovering the roots of given powers, or quantities.

The rule for Simple Quantities is,

Divide the exponent, or exponents, of the given quantity by the index of that power whose root is required. This follows from Involution, Art. 100.

Thus, to find the 3*d*. or cube root of $64a^6$?

The exponent of 64 being 1, we have $64^{\frac{1}{3}}a^{\frac{6}{3}}$, or $64^{\frac{1}{3}}a^2$, or $4a^2$ the root required: for $64^{\frac{1}{3}} = 4$. And $4a^2 \times 4a^2 \times 4a^2 = 64a^6$.

And the cube root of $-64a^6$ is $-4a^2$.

Therefore the root of the product of two or more powers is the product of their roots; for $4a^2 = 4 \times a^2$; and the square root $= 2 \times a$.

Also the cube root of $10x^3$ is $10^{\frac{1}{3}}x^{\frac{3}{3}}$ or $10^{\frac{1}{3}}x$. The coëfficient $10^{\frac{1}{3}}$ is called a *surd*, because 10 will not admit of an exact root.

104. An even root of an affirmative quantity may be either + or −;

Thus the square root of a^2 is $+a$, or $-a$; for $+a \times +a = a^2$; and $-a \times -a = a^2$.

But any even root of a negative quantity is impossible:

Thus the 2*d*. or square root of $-a^2$ is impossible, for $+a \times +a$, nor $-a \times -a$ will produce $-a^2$. In this case the root is represented thus, $\sqrt{(-a^2)}$, or thus $(-a^2)^{\frac{1}{2}}$. Also $(b + x^2)^{\frac{2}{3}}$ denotes the 3*d*. or cube root of $(b + x^2)^2$. And $(a^2 - xy)^{\frac{1}{n}}$ the *n*th. root of $a^2 - xy$.

105. An odd root of any quantity will have the same sign as the quantity itself: thus the 3*d*. root of $+64a^6$ is $+4a^2$; and the 3*d*. root of $-64a^6$ is $-4a^2$, as above.

106. The root of a fraction is found by taking the roots of the numerator and denominator:

Thus the cube root of $\frac{8x^3}{27z^3}$ is $\frac{2x}{3z}$. And the *n*th. root of $\frac{a^2}{b^2}$ is $\frac{a^{\frac{2}{n}}}{b^{\frac{2}{n}}}$. Also, the square root of $\frac{a^{-2}}{b^{-2}}$ is $\pm\frac{a^{-1}}{b^{-1}}$.

If the numerator of the fraction be 1, the root may be denoted by the root of the denominator with a contrary sign to the index:

Thus, the cube root of $\frac{1}{x^3}$ is $\frac{1}{x}$ or x^{-1}: For $\frac{1}{x}=\frac{x}{x^2}=x^{1-2}=x^{-1}$. (71. *Examp*. 5.)

To extract the Square Root of a Compound Quantity.

107. In order to discover the rule for this purpose, it may be necessary to consider the formation of a square.

The square of a binomial $x+a$ is $x^2+2ax+a^2$, or x^2+a^2+ twice the product of the roots x and a;

of a trinomial $x+a+c$, is $(x+a)^2+c^2+$ twice the product of the roots (considering $x+a$ as making one term):

of a quadrinomial $x+a+c+d$, is $(x+a+c)^2+d^2+$ twice the product of the roots; &c. &c.

Therefore $(x+a+c)^2=(x+a)^2+(x+a)\,c\times 2+c^2$

or $x^2+2ax+a^2+2cx+2ca+c^2$, which consists of the three following products;

viz. $x\times x$
$(2x+a)\,a$
$(2x+2a+c)\,c$:

Consequently if x^2 the first term in the square, be divided by x, the quotient is x the first term in the root:

And $2ax+a^2$ divided by $2x+a$ gives a the second term in the root.

Also the remainder $2cx+2ca+c^2$ divided by $2x+2a+c$ gives c the third term.

Again,

$$(x+a+c+d)^2=x^2+2ax+a^2+2cx+2ca+c^2+(x+a+c)d\times 2+d^2,$$

$$\text{or } x^2+2ax+a^2+2cx+2ca+c^2+2dx+2da+2dc+d^2,$$

which is made up of the 4 products

$x\times x$
$(2x+a)\ a$
$(2x+2a+c)\ c$
$(2x+2a+2c+d)\ d.$

Now the three first terms of the root are found as in the preceding trinomial; and $2x+2a+2c+d$ is the divisor which gives the *4th* term d.

And here we may observe that the terms in the root are constantly doubled, and the next term added, to form the divisors: Hence the following rule,

Range the quantities according to the dimensions of some letter, and set the root of the first term in the quotient:

Subtract the square of the root thus found from the first term; and bring down the two next terms for a dividend:

Divide the first term of the dividend by double the root, and set the result in the quotient, and also in the divisor:

Multiply the divisor thus augmented by the last result, and subtract the product from the dividend: then proceed as before, till all the terms are brought down, or the root extracted as far as may be thought necessary.

Examples.

1. To find the square root of $x^2+2ax+a^2+2cx+2ca+c^2+2dx+2da+2dc+d^2$.

$$x^2+2ax+a^2+2cx+2ca+c^2+2dx+2da+2dc+d^2.\ (x+a+c+d, \text{ the root.}$$

$$x^2$$

$$2x+a)\ 0\quad 2ax+a^2$$

$$2ax+a^2$$

$$2x+2a+c)\ 0\quad 2cx+2ca+c^2$$

$$2cx+2ca+c^2$$

$$2x+2a+2c+d)\ 0\quad 2dx+2da+2dc+d^2$$

$$2dx+2da+2dc+d^2$$

$$0$$

Hence the rule for extracting the square root of a number is immediately derived: and if the root consists of 4 figures, x will stand for thousands, a for hundreds, c for tens, and d for units.

2. To extract the square root of $4a^8-16a^6-16a^5+12a^4+32a^3+24a^2+8a+1$.

$$4a^8-16a^6-16a^5+12a^4+32a^3+24a^2+8a+1\ (2a^4-4a^2-4a-1, \text{ root.}$$

$$4a^8$$

$$4a^4-4a^2)\ 0-16a^6-16a^5$$

$$-16a^6+16a^4$$

$$-16a^5-16a^4+12a^4$$

$$4a^4-8a^2-4a)\ \text{or } -16a^5-4a^4+32a^3+24a^2$$

$$-16a^5+32a^3+16a^2$$

$$4a^4-8a^2-8a-1)\ -4a^4+8a^2+8a+1$$

$$-4a^4+8a^2+8a+1$$

$$0$$

3. To find the square root of $a^2x^2+a^2$.

$a^2x^2+a^2=a^2(x^2+1)$, therefore we have to approximate the root of x^2+1:

$$x^2\ 1\ \left(x+\frac{1}{2x}-\frac{1}{8x^3}+\frac{1}{16x^3}-\frac{5}{128x^7}, \text{ \&c. root.}\right.$$

$$x^2$$

$$2x+\frac{1}{2x})\ +1$$

$$+1+\frac{1}{4x^2}$$

$$2x+\frac{1}{x}-\frac{1}{8x^3})\ -\frac{1}{4x^2}$$

$$-\frac{1}{4x^2}-\frac{1}{8x^4}+\frac{1}{64x^6}$$

$$2x+\frac{1}{x}-\frac{1}{4a^3}+\frac{1}{16x^5})\ +\frac{1}{8x^4}-\frac{1}{64x^6}$$

$$+\frac{1}{8x^4}+\frac{1}{16x^6}-\frac{1}{64x^8}+\frac{1}{16^2x^{10}}$$

$$-\frac{5}{64x^6}+\frac{1}{64x^8}-\frac{1}{16^2x^{10}}$$

&c. &c.

To extract the Cube Root of a Compound Quantity.

108. We shall derive the rule from the formation of a Cube.

$$(a+c)^3 = a^3 + 3a^2c + 3ac^2 + c^3:$$

$$\text{or } a^3 + \overline{3(a)^2 + 3(a)c + c^2} \times c.$$

And $(a+c+d)^3 = (a+c)^3 + \overline{3(a+c)^2 + 3(a+c)d + d^2} \times d$, (by considering $a+c$ as making one term of a binomial):

And so on, for the cube of any multinomial.

Hence

$$(a+c)^3 = \begin{cases} a^3 \\ +(3a^2+3ac+c^2)c \end{cases}$$

$$(a+c+d)^3 = \begin{cases} a^3 \\ +(3a^2+3ac+c^2)c \\ +(3a^2+6ac+3c^2+3ad+3cd+d^2)\,d, \text{ \&c.} \end{cases}$$

(From the preceding *formulæ*, the rule, *art.* 117. *Arith.* is readily derived)

Or

$$(a+c)^3 = \begin{cases} a^3 \\ +3a^2c+3ac^2+c^3 \end{cases}$$

$$(a+c+d)^3 = \begin{cases} a^3 \\ +3a^2c+3ac^2+c^3 \\ +3a^2d+6acd+3c^2d+3ad^2+3cd^2+d^3 \end{cases}$$

Hence, in the cube of the trinomial $a+c+d$ it appears, that the 2*d.* term $3a^2c$ divided by $3a^2$ (three times the square of the root a) gives c the 2*d.* term in the root: and when the cube of the two first terms $a+c$ is subtracted, the next term $(3a^2d)$ in the remainder divided by the same divisor $(3a^2)$ gives d the 3*d.* term of the root, &c. whence the following Rule,

Arrange the terms according to the dimensions of some letter, as in extracting the square root, and set the root of the first term in the quotient, and subtract its cube from the quantity whose root is required:

Divide the first term of the remainder by 3 times the square of the root, and the quotient is the second term in the root:

Subtract the cube of the root already found from the given quantity, and divide the first term of the remainder by 3 times the square of the first term, as before, and the quotient is the third term in the root:

Subtract the cube of this augmented root from the proposed quantity; and proceed as before, should there be any remainder.

Examples.

1. To find the cube root of $a^3+3a^2c+3ac^2+c^3+3a^2d+6acd+3c^2d+3ad^2+3cd^2+d^3$.

$$a^3+3a^2c+3ac^2+c^3+3a^2d+6acd+3c^2d+3ad^2+3cd^2+d^3, \quad (a+c+d. \text{ root.}$$

$$\underline{a^3}$$

$$3a^2)\ 0 \quad +3a^2c \ldots \text{ first term of the remainder.}$$

$$a^3+3a^2c+3ac^2+c^3+3a^2d+6acd+3c^2d+3ad^2+3cd^2+d^3.$$

$$(a+c)^3 = \underline{a^3+3a^2c+3ac^2+c^3}$$

$$0 \qquad 3a^2) \quad \underline{+3a^2d} \ldots\ldots \text{first term of remainder.}$$

Now $(a+c+d)^3$ will be the quantity proposed, and consequently $a+c+d$ is the root required.

2. To extract the cube root of x^3+a^3

$$x^3+a^3 \quad \left(x+\frac{a^3}{3x^2}-\frac{a^6}{9x^5}, \text{ \&c. root.}\right.$$

$$\underline{x^3}$$

$$3x^2) \quad +a^3 \ldots\text{.first term of the remainder.}$$

$$\left(x+\frac{a^3}{3x^2}\right)^3 = \begin{array}{l} x^3+a^3 \\ \underline{x^3+a^3+\dfrac{a^6}{3a^3}+\dfrac{a^9}{27a^6}} \end{array}$$

$$3x^2) \quad -\frac{a^6}{3a^3}-\frac{a^9}{27a^0}$$

&c.

109. This rule adapted to the extraction of a cube root in numbers, will be as follows;

Point the number into periods of three figures each (beginning at the units) and find the greatest cube in the first period on the left hand, and set its root in the quotient for the first figure of the required root:

Subtract the cube from the period above it, and bring down the next period to the remainder for a dividend:

Divide the dividend by 3 times the square of the root, and the first quotient figure will be the second figure in the root:

Subtract the cube of the root from the two first periods on the left hand, and to the remainder bring down the next period for a new dividend:

Divide this dividend by 3 times the square of the root, and the first quotient figure is the third figure in the root:

Subtract the cube of the root from the three left hand periods; then proceed as before till all the periods are brought down.

N. B. In dividing by 3 times the square, it is only the first quotient figure that is wanting, and therefore it will not be necessary to continue the division.

3. To extract the cube root of 269210725993.

$$\begin{array}{rl}
 & 26\dot{9}21\dot{0}72\dot{5}99\dot{3}\ \ (6457\ \textit{root.} \\
 & 216 \\
6^2\times3=108) & \overline{53210}\ (4 \\
 & \overline{269210} \\
64^3\ldots= & 262144 \\
64^2\times3=12288) & \overline{7066725}\ (5 \\
 & \overline{269210725} \\
645^3= & 268336125 \\
645^2\times3=1248075) & \overline{874600993}\ (7 \\
 & \overline{269210725993} \\
6457^3= & \underline{269210725993} \\
 & 0
\end{array}$$

This rule is easily remembered; and may be made general by changing the indices: Thus if the *m*th. root is required; then, instead of the 3*d*. root, and 3*d*. power, we must take the *m*th. root, and *m*th. power; and for 3 times the square, make use of *m* times the *m*—1th power: &c.

But in extracting the higher roots of numbers, the divisors frequently give the quotient figures too great; the true figure, however, is found by a trial or two: Thus to extract the 5*th* root of 5559251349024.

$$\begin{array}{rl}
 & 55\dot{5}9251\dot{3}4902\dot{4}\ \text{(354 \textit{root}.} \\
3^5 = & \underline{243} \\
3^4 \times 5 = 405) & \underline{31292513}\ (5 \\
 & 55592513\ \text{... two first periods} \\
35^5 = & \underline{52521875} \\
35^4 \times 5 = 7503125) & \underline{307063849024}\ (4 \\
354^5 = & 5559251349024\text{, therefore 354 is the \textit{root}.}
\end{array}$$

The second divisor 405 will give 7 for the second quotient figure, but 6 is too great; and 5 the true figure is found by raising 35 to the 5*th*. power.

In Algebra, when the proposed quantity is an exact power, its root may frequently be found by inspection: Thus in *Ex*. 1. we find the three cubes a^3, c^3, and d^3 in the given quantity; and the three roots connected with their proper signs, is the root required.

Of Dr. Halley's rational Formulæ for the Roots of pure Powers.

110. LET a^2+b be a quantity whose *square root* is required: where b is supposed to be small when compared with a^2.

Put $a+x=(a^2+b)^{\frac{1}{2}}$. Then squaring both sides (75. *Ax*. 6.) we have $a^2+2ax+x^2=a^2+b$; and $2ax+x^2=b$; whence $x=\dfrac{b}{2a+x}$.

But if b is small when compared with a^2, then x^2 will be much less than $2ax$, and consequently $2ax$ may be taken for the value of b, nearly: therefore if we reject x^2, in the equation $2ax + x^2 = b$*, we have $2ax = b$, and $x = \frac{b}{2a}$ for the first approximate value of x; which being substituted for x in the fraction $\frac{b}{2a+x}$ gives $x = \frac{b}{2a + \frac{b}{2a}} = \frac{ab}{4a^2+b}$ the second approximation; therefore the root $a + x = a + \frac{ab}{4a2+b} = (a^2+b)^{\frac{1}{2}}$.

Again. Let the *cube root* of $a^3 + b$ be required; (b being supposed small, as before): and put $a + x = (a^3 + b)^{\frac{1}{3}}$:

Then $a^3 + 3a^2x + 3ax^2 + x^3 = a^3 + b$;

whence $3a^2x + 3ax^2 + x^3 = b$; and rejecting x^3 on account of its smallness, we get $3a^2x + 3ax^2 = b$, whence $x = \frac{b}{3a^2+3ax}$.

Now in the equation $3a^2x + 3ax^2 = b$, the term $3ax^2$ is supposed to be small when compared with $3a^2x$, therefore *that* being rejected, we have $3a^2x = b$, and $x = \frac{b}{3a^2}$ the first approximate value of x in this case, which put for x in the fraction $\frac{b}{3a^2+3ax}$, gives $x = \frac{b}{3a^2 + \frac{3ab}{3a^2}}$ the second approximation;

Whence the root $a + x = a + \frac{ab}{3a^3+b} = (a^3+b)^{\frac{1}{3}}$.

And if $a + x = (a^4 + b)^{\frac{1}{4}}$, then (omitting all the terms in which x is above the $2d.$ power) $a + x$ will be $= a + \frac{ab}{4a^4 + \frac{3}{2}b} = (a^4 + b)^{\frac{1}{4}}$.

$$\text{Hence } (a^2+b)^{\frac{1}{2}} = a + \frac{ab}{2a^2 + \frac{1}{2}b}.$$
$$(a^3+b)^{\frac{1}{3}} = a + \frac{ab}{3a^3+b}.$$
$$(a^4+b)^{\frac{1}{4}} = a + \frac{ab}{4a^4 + \frac{3}{2}b}.$$
$$(a^5+b)^{\frac{1}{5}} = a + \frac{ab}{5a^5+2b}.$$
$$(a^6+b)^{\frac{1}{6}} = a + \frac{ab}{6a^6 + \frac{5}{2}b}.$$
&c. &c.

---

* Let $a = 50$, and $b = 1$ (a small number) then $2ax + x^2 = b$ becomes $100x + x^2 = 1$, in which equation x must be less than $\frac{1}{100}$, and consequently x^2 less than $\frac{1}{10000}$.

When the given quantity is a *residual*, the latter factors in the root will be negative: Thus $(a^6 - b^{\frac{1}{6}} = a - \frac{ab}{6a^6 - \frac{5}{2}b}$. And so of others.

These are the rational formulæ of Dr. Halley (*Philos. Trans.* 1694.) who has informed us however, that M. de Lagney first gave the rule for the cube root. The irrational formulæ are surds derived nearly in the same manner.

111. But the foregoing expressions may be rendered more commodious for practice as follows:

Let N represent the quantity whose root is required, and r its root:

Then because
$$a + \frac{ab}{2a^2 + \frac{1}{2}b} = \frac{4a^3 + 3ab}{4a^2 + b} = \frac{a^2 + 3\ (a^2 + b)}{3a^2 + a^2 + b} \times a$$
$$a + \frac{ab}{3a^3 + b} = \frac{6a^4 + 4ab}{6a^3 + 2b} = \frac{2a^3 + 4\ (a^3 + b)}{4a^3 + 2\ (a^3 + b)} \times a,$$
$$a + \frac{ab}{4a^4 + \frac{3}{2}b} = \frac{8a^5 + 5ab}{8a^4 + 3b} = \frac{3a^4 + 5\ (a^4 + b)}{5a^4 + 3\ (a^4 + b)} \times a,$$
&c. &c. &c.

If N be substituted for $a^2 + b$, $a^3 + b$, &c.

we have
$$\frac{a^2 + 3\ (a^2 + b)}{3a^2 + a^2 + b} \times a = \frac{a^2 + 3N}{3a^2 + N} \times a = r.$$
$$\frac{2a^3 + 4\ (a^3 + b)}{4a^3 + 2\ (a^3 + b)} \times a = \frac{2a^3 + 4N}{4a^3 + 2N} \times a = r.$$
$$\frac{3a^4 + 5\ (a^4 + b)}{5a^4 + 3\ (a^4 + b)} \times a = \frac{3a^4 + 5N}{5a^4 + 3N} \times a = r.$$
&c. &c.

Which converted into analogies (88. *cor.* 3.) will be

$3a^2 + N : a^2 + 3N :: a : r.$
$4a^3 + 2N : 2a^3 + 4N :: a : r.$
$5a^4 + 3N : 3a^4 + 5N :: a : r.$
&c.

And since it appears that the numeral coefficients are constantly 1 greater, and 1 less than the *index* of the power whose root is to be extracted, the law of continuation is manifest: Therefore, putting n for that index, we have the following general Rule:

$(n + 1)\ a^n + (n - 1)\ N : (n - 1)\ a^n + (n + 1)\ N :: a : r$ (118. Arith.)

For, suppose the 4th. root is required;

Then, in the proportion $5a^4 + 3N : 3a^4 + 5N :: a : r$,
n is $= 4$
the cofficients 5 and 3...... are $n + 1$ and $n - 1$;
and $N = a^4 + b$, where b is small when compared with a^4.

If n is an odd number, $\frac{n+1}{2}$ and $\frac{n-1}{2}$ will be whole numbers which may be used instead of $n + 1$ and $n - 1$, as in extracting the cube root. (118. Arith. *Ex.* 1.)

This method, when applied to the extraction of the higher roots of numbers, is the most expeditious of any, if we except that by Logarithms. See Arith. *Art.* 118, &c.

OF SURDS.

112. SURDS or Radical Quantities are such as have no exact root. The roots however, are designated by means of the radical sign $\sqrt{\ }$, or by fractional indices.

Thus, the square root of 5 is expressed by $\sqrt{5}$, or $5^{\frac{1}{2}}$.

Also $\sqrt[2]{5^3}$, or $5^{\frac{3}{2}}$ denotes the square root of the cube of 5.

And $\sqrt[5]{(a+b)}$, or $(a+b)^{\frac{1}{5}}$ the 5*th*. root of $a+b$: &c. (104.)

113. A rational quantity may be exhibited under various surd forms:

Thus, taking the number 6, for example;

Then $6 = \sqrt{36} = \sqrt{(6 \times 6)} = \sqrt{(4 \times 9)} = \sqrt{(3 \times 12)} = \sqrt{(2 \times 18)} = \sqrt{(1 \times 36)}$
$= \sqrt{4} \times \sqrt{9} = 2\sqrt{9} = 3\sqrt{4} = \sqrt{3} \times \sqrt{12} = \sqrt{6} \times \sqrt{6} = 216^{\frac{1}{3}}$, &c.

And if the quantity be a,

We have $a = \sqrt{a^2} = \sqrt{(a \times a)} = \sqrt{(\frac{1}{4}a \times 4a)} = \sqrt{a} \times \sqrt{a} = \sqrt{a} \times \sqrt{\frac{a^2}{a}}$
$= a^{\frac{1}{3}} \times a^{\frac{2}{3}} = a^{\frac{m}{m}} = a^{\frac{1}{m}} \times a^{\frac{m-1}{m}} = \left(a^{\frac{1}{m}}\right)^m = \left(a^m\right)^{\frac{1}{m}}$, and innumerable other expressions, which will be evident if we consider that the square, or cube, &c. root of any quantity when squared, or cubed, &c. must give the quantity itself.

114. Hence, to bring a rational quantity to the form of a square, or a cube, &c. root, raise it to the 2*d*. or 3*d*. &c. power, and set this quantity under the index denoting the root.

Thus 4 under the forms of the 2*d*, 3*d*, 4*th*, and $\frac{2}{3}$ roots, will be $16^{\frac{1}{2}}$, $64^{\frac{1}{3}}$, $256^{\frac{1}{4}}$, $8^{\frac{2}{3}}$.

Also $a^{\frac{1}{3}}$ reduced to the form of the square root is $\left(a^{\frac{2}{3}}\right)^{\frac{1}{2}}$.

And a^{r} reduced to the *m*th. root is $\left(a^{rm}\right)^{\frac{1}{m}}$.

Generally: $a^{\frac{r}{s}}$ reduced to the form of the $\frac{n}{m}$th. root is $\left(a^{\frac{r}{s}\times\frac{m}{n}}\right)^{\frac{n}{m}}$,

or $\left(a^{\frac{rm}{sn}}\right)^{\frac{n}{m}}$.

For if the multiplication by $\frac{m}{n}$ involves to the power $\frac{m}{n}$, the multiplication by its reciprocal $\frac{n}{m}$ must reduce it again to the root.

115. To reduce quantities with different indices, to other equivalent ones having a common index.

Reduce the indices to a common denominator; then involve each quantity to the power denoted by its numerator.

Examples.

1. Reduce $8^{\frac{1}{3}}$ and $9^{\frac{1}{2}}$ to equivalent quantities having a common index,

$\frac{1}{3}=\frac{2}{6}$; and $\frac{1}{2}=\frac{3}{6}$.

Therefore $8^{\frac{1}{3}}=8^{\frac{2}{6}}=64^{\frac{1}{6}}$

$9^{\frac{1}{2}}=9^{\frac{3}{6}}=729^{\frac{1}{6}}$.

And the quantities are $64^{\frac{1}{6}}$ and $729^{\frac{1}{6}}$, having the common index $\frac{1}{6}$.

For $8^{\frac{1}{3}}=2$, and $64^{\frac{1}{3}}=2$. Also $9^{\frac{1}{2}}=3$, and $729^{\frac{1}{6}}=3$.

116. When the quantities are to be reduced to equivalent ones having a *given index*, it may be done by the general form in the preceding article: thus,

2. Let $8^{\frac{1}{3}}$ and $9^{\frac{1}{2}}$ be reduced to equivalent quantities having the common index $\frac{1}{4}$.

$$\text{Then } \left(8^{\frac{1}{3}\times\frac{4}{1}}\right)^{\frac{1}{4}} = \left(8^{\frac{4}{3}}\right)^{\frac{1}{4}} = \left(4096^{\frac{1}{3}}\right)^{\frac{1}{4}} = 16^{\frac{1}{4}}.$$

$$\text{And } \left(9^{\frac{1}{2}\times\frac{4}{1}}\right)^{\frac{1}{4}} = \left(9^{\frac{4}{2}}\right)^{\frac{1}{4}} = \left(9^{2}\right)^{\frac{1}{4}} = 81^{\frac{1}{4}}.$$

Ans. $16^{\frac{1}{4}}$ and $81^{\frac{1}{4}}$.

3. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{2}{3}}$ to the common index $\frac{1}{6}$.

$$\left(3^{\frac{1}{2}\times\frac{6}{1}}\right)^{\frac{1}{6}} = \left(3^{3}\right)^{\frac{1}{6}} = 27^{\frac{1}{6}}.$$

$$\left(2^{\frac{2}{3}\times\frac{6}{1}}\right)^{\frac{1}{6}} = \left(2^{4}\right)^{\frac{1}{6}} = 16^{\frac{1}{6}}.$$

Ans. $27^{\frac{1}{6}}$ and $16^{\frac{1}{6}}$.

4. Let $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, and $c^{\frac{1}{4}}$ be reduced to the common index $\frac{2}{3}$.

$$\left(a^{\frac{1}{2}\times\frac{3}{2}}\right)^{\frac{2}{3}} = \left(a^{\frac{3}{4}}\right)^{\frac{2}{3}};\ \left(b^{\frac{1}{3}\times\frac{3}{2}}\right)^{\frac{2}{3}} = \left(b^{\frac{1}{2}}\right)^{\frac{2}{3}};\ \left(c^{\frac{1}{4}\times\frac{3}{2}}\right)^{\frac{2}{3}} = \left(c^{\frac{3}{8}}\right)^{\frac{2}{3}}.$$

Ans. $\left(a^{\frac{3}{4}}\right)^{\frac{2}{3}}$, $\left(b^{\frac{1}{2}}\right)^{\frac{2}{3}}$, $\left(c^{\frac{3}{8}}\right)^{\frac{2}{3}}$.

5. Reduce $a^{\frac{r}{n}}$ and $b^{\frac{r}{m}}$ to a common index.

$$\frac{r}{n} = \frac{mr}{nm};\ \text{and } \frac{r}{m} = \frac{nr}{nm}.$$

$$\text{Therefore } a^{\frac{r}{n}} = a^{\frac{mr}{nm}} = \left(a^{m}\right)^{\frac{r}{nm}}.$$

$$\text{and } b^{\frac{r}{m}} = b^{\frac{nr}{nm}} = \left(b^{n}\right)^{\frac{r}{nm}}.$$

Ans. $\left(a^{m}\right)^{\frac{r}{nm}}$, and $\left(b^{n}\right)^{\frac{r}{nm}}$.

Of Multiplying Surd quantities together.

117. It appears from Art. 113. that the product of surds having a common index, is the product of the quantities themselves with the same common index.

Thus $\sqrt{3}\times\sqrt{12}=\sqrt{36}$: or $3^{\frac{1}{2}}\times 12^{\frac{1}{2}}=36^{\frac{1}{2}}$.

And $\sqrt{a}\times\sqrt{a}=\sqrt{a^2}$: or $a^{\frac{1}{2}}\times a^{\frac{1}{2}}=(a^2)^{\frac{1}{2}}$.

Also $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$: or $a^{\frac{1}{2}}\times b^{\frac{1}{2}}=(ab)^{\frac{1}{2}}$.

And since $2\sqrt{9} \times 3\sqrt{4} = 2 \times 3\sqrt{(9 \times 4)} = 6\sqrt{36}$;

Therefore if surds have coefficients their product must be prefixed:

Thus $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$.

And $a^n \left(b^{\frac{1}{m}}\right) \times c^r \left(d^{\frac{1}{m}}\right) = a^n c^r (bd)^{\frac{1}{m}}$, or $a^n c^r b^{\frac{1}{m}} d^{\frac{1}{m}}$.

118. The product of like quantities in the form of surds with the same, or different indices, is found by adding those indices together (45):

Thus $64^{\frac{1}{2}} \times 64^{\frac{1}{3}} = 64^{\frac{1}{2}+\frac{1}{3}} = 64^{\frac{5}{6}}$.

And $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

Also $(a+b)^{\frac{1}{n}} \times (a+b)^{\frac{1}{m}} = (a+b)^{\frac{1}{n}+\frac{1}{m}} = (a+b)^{\frac{n+m}{nm}}$

119. If surds have different indices, reduce them to the same index (115); then find the product as in the preceding articles.

Thus, to find the product of $8^{\frac{1}{3}}$ and $9^{\frac{1}{2}}$:

$\frac{1}{3} = \frac{2}{6}$; and $\frac{1}{2} = \frac{3}{6}$:

Then $8^{\frac{1}{3}} = 8^{\frac{2}{6}} = 64^{\frac{1}{6}}$:

and $9^{\frac{1}{2}} = 9^{\frac{3}{6}} = 729^{\frac{1}{6}}$

Therefore $64^{\frac{1}{6}} \times 729^{\frac{1}{6}} = (64 \times 729)^{\frac{1}{6}} = 46656^{\frac{1}{6}}$, the answer.

Other Examples.

1. Required the product of $6\sqrt{10}$ and $10\sqrt{6}$.

$6 \times 10\sqrt{(10 \times 6)} = 60\sqrt{60}$. *Ans.*

2. What is the product of $5\,(9)^{\frac{1}{3}}$ and $2\,(6)^{\frac{1}{3}}$?

$5 \times 2\,(9 \times 6)^{\frac{1}{3}} = 10\,(54)^{\frac{1}{3}}$. *Ans.*

3. Required the product of $\frac{1}{2}\left(\frac{4}{5}\right)^{\frac{1}{3}}$ and $\frac{2}{3}\left(\frac{5}{6}\right)^{\frac{1}{3}}$?

$\frac{1}{2} \times \frac{2}{3}\left(\frac{4}{5} \times \frac{5}{6}\right)^{\frac{1}{3}} = \frac{1}{3}\left(\frac{4}{6}\right)^{\frac{1}{3}} = \left(\frac{4}{162}\right)^{\frac{1}{3}} = \left(\frac{2}{81}\right)^{\frac{1}{3}}$. *Ans.*

4. Required the product of $a\,(a+\sqrt{c})^{\frac{1}{2}}$ and $b\,(a\,\sqrt{c})^{\frac{1}{2}}$?

$$ab \times \overline{(a+\sqrt{c})\ (a-\sqrt{c})}^{\frac{1}{2}} = ab\ (a^2-c)^{\frac{1}{2}}.\quad \textit{Ans.}$$

5. Required the product of $(x-y)^{-2}$ and $(x-y)^{2}$?

$$(x-y)^{-2} \times (x-y)^{2} = (x-y)^{-2+2} = x-y.\quad \textit{Ans.}$$

6. What is the product of $(x-y)^{\frac{1}{2}}$ and $\dfrac{x}{(x^2-y^2)^{\frac{1}{2}}}$?

$$(x-y)^{\frac{1}{2}} \times \frac{x}{(x^2-y^2)^{\frac{1}{2}}} = x\left(\frac{x-y}{x^2-y^2}\right)^{\frac{1}{2}} = \frac{x}{(x+y)^{\frac{1}{2}}}.\quad \textit{Ans.}$$

7. What is the product of $(a-x)^{\frac{1}{3}}$ and $x^{\frac{1}{2}}$?

$$(a-x)^{\frac{1}{3}} = (a-x)^{\frac{2}{6}} = (a^2-2ax+x^2)^{\frac{1}{6}}$$

$$x^{\frac{1}{2}} = x^{\frac{3}{6}} = (x^3)^{\frac{1}{6}}$$

$$(a^2-2ax+x^2)^{\frac{1}{6}} \times (x^3)^{\frac{1}{6}} = (a^2x^3-2ax^4+x^5)^{\frac{1}{6}}.\quad \textit{Ans.}$$

8. Required the product of $\sqrt{(-a)}$ and $\sqrt{(-a)}$?

$\sqrt{(-a)} \times \sqrt{(-a)} = \sqrt{(-a \times -a)} = \sqrt{a^2}$: but the square root of a^2 is $+a$, or $-a$.

Or thus: $(-a)^{\frac{1}{2}} \times (-a)^{\frac{1}{2}} = (-a)^{\frac{1}{2}+\frac{1}{2}} = (-a)^1 = -a$.

This last method shews that the square of $\sqrt{(-a)}$ is $-a$; but it does not prove that it is not $+a$. Now the preceding operation gives both $+a$, and $-a$, conformable to the rules of multiplication, and the extraction of roots: for if $\sqrt{(-a)}$ be a negative root, its square by actual multiplication will be positive, and this positive square will have a positive, and a negative root. But it may be said that $\sqrt{(-a)}$ denotes the root of a negative, not a negative root; this objection however, is obviated by the process; for $\sqrt{(-a)} \times \sqrt{(-a)}$ or $\sqrt{(-a \times -a)}$, and $-\sqrt{a} \times -\sqrt{a}$, both give $+\sqrt{a^2}$. It seems therefore not more repugnant to algebraic method, in making the square of $\sqrt{(-a)}$ equal to $+a$ or $-a$, than in admitting $+a$, or $-a$ to be the square root of $+a^2$.

Hence it appears that the square of $\sqrt{(-a)}$ is $\pm a$.
of $\sqrt{(-a^2)}$ is $\pm a^2$.
of $1+\sqrt{(-2)}$ is $1+\sqrt{(-8)} \pm 2$.
And that the product of $\sqrt{(-a)}$ and $\sqrt{(-b)}$ is $\pm\sqrt{ab}$.

Division of Surds.

120. Reduce the surds to the same index; then the quotient of the surd quantities, with that index, will be the answer.

Examples.

1. Divide $\sqrt{18}$ by $\sqrt{2}$.

$$\frac{\sqrt{18}}{\sqrt{2}}=\sqrt{\frac{18}{2}}=\sqrt{9}=3. \textit{ Ans.}$$

2. Divide $46656^{\frac{1}{6}}$ by $8^{\frac{1}{3}}$?

$$8^{\frac{1}{3}}=8^{\frac{2}{6}}=64^{\frac{1}{6}}:$$

$$\frac{46656^{\frac{1}{6}}}{66^{\frac{1}{6}}}=\left(\frac{46656}{64}\right)^{\frac{1}{6}}=729^{\frac{1}{6}}=3. \textit{ Ans.}$$

3. Divide $\sqrt{a}$ by $\sqrt{b}$.

$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}. \textit{ Ans.}$$

4. Divide $a^{\frac{1}{n}}$ by $b^{\frac{1}{m}}$.

$$a^{\frac{1}{n}}=a^{\frac{m}{nm}}=(a^m)^{\frac{1}{nm}}: \text{ and } b^{\frac{1}{m}}=b^{\frac{n}{nm}}=(b^n)^{\frac{1}{nm}}.$$

$$\frac{(a^m)^{\frac{1}{nm}}}{(b^n)^{\frac{1}{nm}}}=\left(\frac{a^m}{b^n}\right)^{\frac{1}{nm}}. \textit{ Ans.}$$

121. If the surds have coefficients their quotient must be prefixed. And the quotient of like surds is found by subtracting the index of the divisor from that of the dividend. (49).

5. Divide $8\sqrt{10}$ by $4\sqrt{2}$.

$$\frac{8\sqrt{10}}{4\sqrt{2}}=2\sqrt{\tfrac{10}{2}}=2\sqrt{5}. \textit{ Ans.}$$

6. Divide $\frac{1}{2}\sqrt{\frac{5}{8}}$ by $\frac{2}{5}\sqrt{\frac{5}{6}}$?

$$\frac{\frac{1}{2}\sqrt{\frac{5}{8}}}{\frac{2}{5}\sqrt{\frac{5}{6}}}=\tfrac{5}{4}\sqrt{\tfrac{3}{4}}=\frac{4\times5}{4\times4}\sqrt{\tfrac{3}{4}}=\tfrac{5}{16}\sqrt{\frac{3\times16}{4}}=\tfrac{5}{16}\sqrt{12}. \textit{ Ans.}$$

7. Divide $a\sqrt{b}$ by $c\sqrt{d}$?

$$\frac{a\sqrt{b}}{c\sqrt{d}}=\frac{a}{c}\sqrt{\frac{b}{d}}. \textit{ Ans.}$$

8. Divide $ax^{\frac{1}{2}}$ by $bx^{\frac{1}{3}}$?

$$\frac{ax^{\frac{1}{2}}}{bx^{\frac{1}{3}}} = \frac{a}{b}x^{\frac{1}{2}-\frac{1}{3}} = \frac{a}{b}x^{\frac{1}{6}}. \text{ Ans.}$$

9. Divide $(a-)^{\frac{1}{3}}$ by $(a-x)^{\frac{2}{3}}$?

$$\frac{(a-x)^{\frac{1}{3}}}{(a-x)^{\frac{2}{3}}} = (a-x)^{\frac{1}{3}-\frac{2}{3}} = (a-x)^{-\frac{1}{3}} = \frac{1}{(a-x)^{\frac{1}{3}}}. \text{ Ans.}$$

10. Divide $1+\sqrt{(-8)}\pm 2$ by $1+\sqrt{(-2)}$?

$$\sqrt{(-8)} = 2\sqrt{(-2)}:$$

$$\begin{array}{r|l|l} 1+\sqrt{(-2)} & 1+2\sqrt{(-2)}\pm 2 & 1+\sqrt{(-2)} \text{ Quotient.} \\ & \underline{1+\sqrt{(-2)}} & \\ & \sqrt{(-2)}\pm 2 & \\ & \underline{\sqrt{(-2)}\pm 2} & \\ & 0 & \end{array}$$

11. Divide ax^{-n} by bx^{m}.

$$\frac{ax^{-n}}{bx^{m}} = \frac{a}{b}\left(\frac{x^{-n}}{x^{m}}\right) = \frac{a}{b}\left(x^{-n-m}\right) = \frac{a}{b}\left(x^{-(n+m)}\right). \text{ Ans.}$$

Of reducing Surds to their simplest terms.

122. THIS principally consists in resolving the quantity into its component factors, or separating the rational and irrational parts; which being done, the root of the greatest rational power it contains, will be the coefficient of the irrational part.

These reductions are frequently necessary when surds are to be added, or subtracted.

Examples.

1. Reduce $\sqrt[3]{(a^3b^3x)}$ to its simplest terms.

$$\sqrt[3]{(a^3b^3x)} \text{ or } (a^3b^3x)^{\frac{1}{3}} = (a^3b^3)^{\frac{1}{3}} \times x^{\frac{1}{3}} = abx^{\frac{1}{3}}. \text{ Ans.}$$

2. Reduce $\sqrt{363}$ to its simplest terms.

$$\sqrt{363} = \sqrt{(11 \times 11 \times 3)} = 11\sqrt{3}. \text{ Ans.}$$

3. Reduce $\left(\frac{2}{81}\right)^{\frac{1}{3}}$ to its simplest terms.

$$\left(\frac{2}{81}\right)^{\frac{1}{3}}=\left(\frac{2\times9}{81\times9}\right)^{\frac{1}{3}}=\frac{18^{\frac{1}{3}}}{729^{\frac{1}{3}}}=\frac{18^{\frac{1}{3}}}{9}=\frac{1}{9}(18)^{\frac{1}{3}}.\quad \textit{Ans.}$$

4. Reduce $\sqrt{\frac{50}{147}}$ to its most simple terms.

$$\sqrt{\frac{50}{147}}=\sqrt{\frac{25\times2}{49\times3}}=\frac{5}{7}\sqrt{\frac{2}{3}}=\frac{5\times3}{7\times3}\sqrt{\frac{2}{3}}=\frac{5}{21}\sqrt{\frac{2\times9}{3}}=\frac{5}{21}\sqrt{6}.\quad \textit{Ans.}$$

5. Reduce $(54ax^3-54ax^4)^{\frac{1}{3}}$ to its simplest terms.

$$(54ax^3-54ax^4)^{\frac{1}{3}}=(27x^3\times2a-27x^3\times2ax)^{\frac{1}{3}}=3x\,(2a-2ax)^{\frac{1}{3}}.\quad \textit{Ans.}$$

Of the Addition and Subtraction of Surds.

123. REDUCE the proposed quantities, by the preceding article, so that the surd parts are the same (if they admit of such reduction); then denote the sum, or difference, by means of their coefficients.

Examples.

1. Required the sum, and difference of $\sqrt{a^2b}$ and $\sqrt{c^2b}$.

$\sqrt{a^2b}=a\sqrt{b}$: and $\sqrt{c^2b}=c\sqrt{b}$.

Therefore $(a+c)\sqrt{b}$, is the *sum*:

And $(a\backsim c)\sqrt{b}$, the *difference*.

2. Required the sum, and difference of $\sqrt{245}$ and $\sqrt{605}$.

$\sqrt{245}=\sqrt{(49\times5)}=7\sqrt{5}$. And $\sqrt{605}=\sqrt{(121\times5)}=11\sqrt{5}$.

Hence $18\sqrt{5}$, the *sum*. And $4\sqrt{5}$ the *difference*.

3. What is the sum, and also the difference of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$?

$$\sqrt{\frac{2}{3}}=\frac{3}{3}\sqrt{\frac{2}{3}}=\frac{1}{3}\sqrt{\frac{2\times9}{3}}=\frac{1}{3}\sqrt{6}.$$

And $\sqrt{\frac{27}{50}}=\sqrt{\frac{9\times3}{25\times2}}=\frac{3}{5}\sqrt{\frac{3}{2}}=\frac{3\times2}{5\times2}\sqrt{\frac{3}{2}}=\frac{3}{10}\sqrt{\frac{3\times4}{2}}=\frac{3}{10}\sqrt{6}.$

$$\frac{1}{3}\sqrt{6}=\frac{10}{30}\sqrt{6}.\quad\text{And }\frac{3}{10}\sqrt{6}=\frac{9}{30}\sqrt{6}.$$

$$\frac{10}{30}\sqrt{6} + \frac{9}{30}\sqrt{6} = \frac{19}{30}\sqrt{6}, \text{ the } \textit{sum}.$$

$$\frac{10}{30}\sqrt{6} - \frac{9}{30}\sqrt{6} - \frac{1}{30}\sqrt{6}, \text{ the } \textit{difference}.$$

4. Required the sum, and difference of $(24a^4x)^{\frac{1}{3}}$ and $(40b^3x)^{\frac{1}{3}}$.

$$(24a^4x)^{\frac{1}{3}} = 24^{\frac{1}{3}}a^{\frac{4}{3}}x^{\frac{1}{3}}. \text{ And } (40b^3x)^{\frac{1}{3}} = 40^{\frac{1}{3}}bx^{\frac{1}{3}}:$$

$$\text{Therefore } (24^{\frac{1}{3}}a^{\frac{4}{3}} + 40^{\frac{1}{3}}b)x^{\frac{1}{3}} \text{ is the } \textit{sum}.$$

$$\text{And } (24^{\frac{1}{3}}a^{\frac{4}{3}} \backsim 40^{\frac{1}{3}}b)x^{\frac{1}{3}} \text{ the } \textit{difference}.$$

Of involving Surds: And extracting their Roots.

124. SURDS are involved by multiplying the index by the exponent of the power to which it is to be raised. (100).

Thus, the cube of $a^{\frac{2}{3}}$ is $a^{\frac{2}{3}\times 3} = a^2$.

And the square of $(a^2-x^2)^{\frac{1}{2}}$ is $(a^2-x^2)^{\frac{1}{2}\times 2} = a^2-x^2$.

Also the *4th* power of $(a-z)^{\frac{1}{2}}$ is $(a-z)^{\frac{1}{2}\times 4} = (a-z)^2 = a^2 - 2az \times z^2$.

If the Surds have coefficients, their powers must be prefixed.

Thus, the cube of $3x^{\frac{1}{3}}$ is $27x$.

And the $\frac{2}{3}$ power of $8x^{\frac{1}{2}}$ is $8^{\frac{2}{3}}x^{\frac{1}{2}\times\frac{2}{3}} = 4x^{\frac{1}{3}}$.

Other Examples.

1. What is the *m*th. power of $ax^{\frac{1}{n}}$? *Ans.* $a^m x^{\frac{m}{n}}$

2. Required the square of $7\,(9^{\frac{1}{3}})$? *Ans.* $49\,(81^{\frac{1}{3}})$.

3. Required the cube of $2^{\frac{1}{3}}$?

$$2^{\frac{1}{2}\times 3} = 2^{\frac{3}{2}} = 8^{\frac{1}{2}} = 2\,(2^{\frac{1}{2}}). \quad \textit{Ans.}$$

4. Required the 4th. power of $\frac{1}{7}\sqrt{7}$?

$$\frac{1}{7}\sqrt{7} = \sqrt{\frac{7}{49}} = \left(\frac{1}{7}\right)^{\frac{1}{2}}; \text{ and } \left(\frac{1}{7}\right)^{\frac{1}{2}\times 4} = \left(\frac{1}{7}\right)^2 = \frac{1}{49}. \quad \textit{Ans.}$$

5. What is the square of $\sqrt{a} + \sqrt{b}$? *Ans.* $a + 2\sqrt{ab} + b$.

6. Required the square, and also the cube of $\sqrt{3}-\sqrt{2}$?

$$\begin{array}{l}
\sqrt{3}-\sqrt{2} \\
\sqrt{3}-\sqrt{2} \\
\hline
\quad 3-\sqrt{6} \\
\qquad -\sqrt{6}+2 \\
\hline
\quad 5-\sqrt{24} \ldots\ldots\ldots \text{ the } \textit{square.} \\
\sqrt{3}-\sqrt{2} \\
\hline
5\sqrt{3}-\sqrt{72} \\
\qquad -5\sqrt{2}+\sqrt{48} \\
\hline
5\sqrt{3}-\sqrt{72}-5\sqrt{2}+\sqrt{48}=9\sqrt{3}-11\sqrt{2}, \text{ the } \textit{cube.}
\end{array}$$

For $\sqrt{72}=6\sqrt{2}$; and $\sqrt{48}=4\sqrt{3}$; whence $4\sqrt{3}+5\sqrt{3}=9\sqrt{3}$; and $6\sqrt{2}+5\sqrt{2}=11\sqrt{2}$.

125. *To find the root of a surd,* divide its index by the index of the root to be extracted. And when the surds have rational coefficients, their roots must be prefixed.

Examples.

1. What is the square root of $a^2\ (b^{\frac{2}{3}})$?

The square root of a^2 is a: and the square root of $b^{\frac{2}{3}}$ is $b^{\frac{2}{3\times2}}=b^{\frac{1}{3}}$; therefore $ab^{\frac{1}{3}}$ is the root required.

2. What is the square root of $49c^2\ (81^{\frac{1}{3}})$?

The square root of $49c^2$ is $7c$.

And $81^{\frac{1}{3\times2}}=81^{\frac{1}{6}}$ or $9^{\frac{1}{3}}$ is the square root of $81^{\frac{1}{3}}$.

Ans. $7c\ (9^{\frac{1}{3}})$.

3. Required the square root of 12^3?

$12^{\frac{3}{2}}=1728^{\frac{1}{2}}$ or $12\ (12^{\frac{1}{2}})$. *Ans.*

4. Required the $4th.$ root of $5x^2y^2$? *Ans.* $5^{\frac{1}{4}}\ (xy)^{\frac{1}{2}}$.

5. What is the *m*th. root of $ax^{\frac{r}{n}}$? *Ans.* $a^{\frac{1}{m}}x^{\frac{r}{mn}}$.

6. What is the cube root of $(a^3c^3x)^{\frac{1}{2}}$?

$(a^3c^3x)^{\frac{1}{2}}=(ac)^{\frac{3}{2}}x^{\frac{1}{2}}$; and $(ac)^{\frac{3}{2\times3}}x^{\frac{1}{2\times3}}=a^{\frac{1}{2}}c^{\frac{1}{2}}x^{\frac{1}{6}}$. *Ans.*

7. What is the $-m$th. root of ax^{-n}?

$$a^{\frac{1}{-m}}x^{\frac{-n}{-m}}=a^{-\frac{1}{m}}x^{\frac{n}{m}}.\quad \textit{Ans.}$$

8. Required the square root of $a-2\sqrt{ab}+b$?

Ans. $\sqrt{a}-\sqrt{b}$, or $\sqrt{b}-\sqrt{a}$.

And the square root of $a+2\sqrt{ab}+b$, is $\sqrt{a}+\sqrt{b}$.

126. The roots in the two last examples are obvious by inspection only; but if numbers are substituted for the letters, the square becomes a binomial: thus, let $5=a$, and $3=b$, then $(\sqrt{5}+\sqrt{3})^2=8+\sqrt{60}$, where the rational part 8 is the sum of the squares of the two surds, and the irrational part $\sqrt{60}$, twice their product. To discover the root in such cases, suppose $a+\sqrt{b}$ is the binomial, and let x and y denote the two members of its root, viz. put $(x+y)^2=a+\sqrt{b}$.

Then x^2+y^2+2xy being $=a+\sqrt{b}$,

we shall have $x^2+y^2=a$, and $2xy=\sqrt{b}$.

And $y^2=a-x^2$. Also $4x^2y^2=b$; whence $y^2=\frac{b}{4x^2}$.

Therefore $a-x^2=\frac{b}{4x^2}$.

And $ax^2-x^4=\frac{b}{4}$, or $x^4-ax^2=-\frac{b}{4}$ (77).

Now if $\frac{a^2}{4}$ be added to each side of this equation, we get

$$x^4-ax^2+\frac{a^2}{4}=\frac{a^2}{4}-\frac{b}{4}=\frac{a^2-b}{4};\ (75.\ \textit{Ax.}\ 1)$$

But $x^4-ax^2+\frac{a^2}{4}$ is a square, its root being $x^2-\frac{a}{2}$:

Therefore $x^2-\frac{a}{2}=\sqrt{\frac{a^2-b}{4}}=\frac{\sqrt{(a^2-b)}}{2}$. (75. *Ax.* 7).

whence $x^2=\frac{a}{2}+\frac{\sqrt{(a^2-b)}}{2}=\frac{a+\sqrt{(a^2-b}}{2}$; and $x=\sqrt{\left(\frac{a+\sqrt{(a^2-b}}{2}\right)}$.

Now y being $=\sqrt{(a-x^2)}$,

we have $y=\sqrt{\left(a-\frac{a+\sqrt{(a^2-b)}}{2}\right)}=\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}$.

Therefore the square root of $a+\sqrt{b}$ is $\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}+\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}$.

And if the surd be a residual, $a-\sqrt{b}$:

Then $\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}-\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}$ is the root.

Examp. Let $12-\sqrt{140}$ be the binomial;

Then the expression becomes $\sqrt{\frac{12+\sqrt{4}}{2}}-\sqrt{\frac{12-\sqrt{4}}{2}}=\sqrt{7}-\sqrt{5}$, the square root of $12-\sqrt{140}$.

The above expressions are general for a binomial: but when applied to surds, it is necessary that $\sqrt{(a^2-b)}$ be *rational* to bring out one or both members of the root in simple quadratic surds.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

127. According to the natural arrangement of parts into which Algebra may be divided, this should immediately have followed the Resolution of Simple Equations, *Art.* 74, &c. But we found some knowledge of equations requisite in the articles on Proportion, the extraction of roots, &c. And an acquaintance with these latter branches frequently becomes necessary in the resolution of Problems, whether they produce simple, or quadratic equations.

128. When a Question is proposed, the student should represent the unknown or required quantity, or quantities, by a letter, or letters, as x, y, z, &c. then let him operate with both the given and unknown quantities, by addition, subtraction, multiplication, &c. according to the conditions and tenor of the question; by which means he will obtain one or more Equations involving the unknown letter or letters.—But on this head, a few examples are preferable to many precepts.

Examples.

1. What number is that to which 7 being added, and the sum divided by 3, gives the quotient 13?

Let x denote the required number:

Then the sum of x and 7 is .... $x+7$

which divided by 3 gives .... $\frac{x+7}{3}$

And this must be $=13$, viz............ $\frac{x+7}{3}=13$

Whence $x=32$, the number sought. See Art. 77. *Ex.* 3.

2. There is a number to which if we add 5, and subtract its double from $\frac{1}{2}$ the sum, the remainder will be equal to the said number divided by 3. Required the number?

Let x represent the number sought:

To which adding 5 gives .... $x+5$

Half of this is .... $\frac{x+5}{2}$

And subtracting double the required number, leaves...... $\frac{x+5}{2}-2x$

Which, by the question is $=\frac{1}{3}$ of that number, viz. $\frac{x+5}{2}-2x=\frac{x}{3}$

Whence $x=1\frac{4}{11}$. Art. 77. *Ex.* 6.

3. What two numbers are those whose sum is 20, and difference 6.

Suppose the less number to be x

Then the greater must be .... $x+6$

Their sum is.. $2x+6$

This sum is $=20$, viz. .... $2x+6=20$

Whence $2x=20-6=14$

And $x=\frac{14}{2}=7$ the *less*.

And $7+6=13$ the *greater*.

Or thus, To find two numbers whose sum is s, and difference d.

Let the less be x; then the greater is $x+d$

And their sum is $2x+d$

Therefore $2x+d=s$

Whence $2x=s-d$, and $x=\frac{1}{2}s-\frac{1}{2}d$ the *less*.

And $\frac{1}{2}s-\frac{1}{2}d+d=\frac{1}{2}s+\frac{1}{2}d$, the *greater*.

Therefore when the sum, and difference, of two numbers are given, half the difference added to, and subtracted from half the sum, will be the *greater* and *less*, respectively.

4. The sum of two numbers is 19, and the difference of this squares 95: What are the numbers?

Put x for the greater number, and y for the less.

Then by the question.................... $x+y=19$

And...................................... $x^2-y^2=95$

Now dividing the second equation by the first, we have $\frac{x^2-y^2}{x+y}=\frac{95}{19}$ (75. *Ax.* 4.)

Or by actual division...................... $x-y=5$

Therefore we have the sum of the two numbers, $x+y=19$

and their difference $x-y=5$

Whence, by the preceding example, $\frac{19}{2}+\frac{5}{2}=12$ the *greater*; and $\frac{19}{2}-\frac{5}{2}$ $=7$ the *less*.

5. What two fractions are those whose sum is 1, and the greater divided by the less gives the quotient 10?

For the less put................................. x

Then the greater will be .... $1-x$

Whence by the question .... $\frac{1-x}{x}=10$

or............ $1-x=10x$

therefore... $1=11x$

And... $\frac{1}{11}=x$, the *less*

And $1-\frac{1}{11}=\frac{10}{11}$ the *greater*.

6. A General having detached 620 men to take possession of a strong post, and $\frac{3}{7}$ of the remainder of his troops to watch the motions of the enemy, finds that he has only $\frac{3}{13}$ of his army left; what was his whole force?

Let the whole number of men be...................... x

Then after 620 were detached the remainder was........... $x-620$

$\frac{3}{7}$ of these are.. $\frac{3x-1860}{7}$

And $\frac{3}{13}$ of the whole is... $\frac{3x}{13}$

Now by the question, the two last parts with

620 must make the whole; viz......... $620+\frac{3x-1860}{7}+\frac{3x}{13}=x$

which cleared of fractions gives $56420+39x-24180+21x=91x$

whence... $56420-24180=31x$

or....... $32240=31x$

and $x=1040$, the *Answer.*

7. Three battalions of unequal force are in column of march; the extent of the first battalion is 216 paces, the extent of the second is equal to that of the first and third together, and the extent of the third is equal to that of the first and half the second: what is the length of the column?

Let the length of the third be x

Then that of the second will be .... $216+x$

The first and half the second together is...... $216+\frac{216+x}{2}$

Which, by the quest. is equal to the third, viz. $216+\frac{216+x}{2}=x$

whence......... $432+216+x=2x$

And........ .. $x=648$ the third

$648+216=864$ the second

216 the first

The whole $=$ 1728 *paces.*

8. The Double Rule of False is founded on the supposition, That the differences between the true and supposed numbers are directly proportional to the respective differences between the true and erroneous results: now it is required to show if the Arithmetical process is conformable to that supposition. (Arith. *Art.* 109.)

Let S and s be the two suppositions, D and d the corresponding errors or differences between the results and the number with which they are compared; also, let x denote the number required.

Then $x-S$, and $x-s$ will be the differences between the true and supposed numbers when the latter are both too little:

And $S-x$, and $s-x$, when they are both too great.

Now by the supposition, $x-S : x-s :: D : d$; whence $Dx-Ds=dx-dS$.

$$\text{and } Dx-dx=Ds-dS$$

$$\text{therefore } x=\frac{Ds-dS}{D-d}.$$

And $S-x : s-x :: D : d$ gives .... $x=\dfrac{dS-Ds}{d-D}$.

Now in both these cases, the errors are alike. And each expression is the *difference* of the products divided by the *difference* of the errors. (*First rule.*)

But when the errors are unlike, we shall have

either $S-x : x-s :: D : d$, whence $x=\dfrac{dS+Ds}{d+D}$:

or $x-S : s-x :: D : d$, and $x=\dfrac{Ds+dS}{D+d}$;

Where the expressions give the *sum* of the products divided by the *sum* of the errors: which is the *second rule.*

9. Divide 10 into three such parts, that when the first is multiplied by 2, the second by 3, and the third by 4, the three products may be equal?

Let x, y, and z, denote the three parts:

Then, by the question............ $x+y+z=10$

and.... $2x=3y=4z$

Now because $2x=4z$, therefore $x=2z$

Also, since $3y=4z$, we have $y=\dfrac{4z}{3}$

Now putting $2z$, and $\dfrac{4z}{3}$ for x and y in the first equation

and we have $2z+\dfrac{4z}{3}+z=10$

whence $6z+4z+3z=30$

and......... $z=\dfrac{30}{13}=2\dfrac{4}{13}$

Or thus. Assume three quantities which being multiplied by 2, 3, and 4, respectively, shall give the same quotient; thus,

Suppose $\dfrac{x}{2}$, $\dfrac{x}{3}$, and $\dfrac{x}{4}$; then $\dfrac{x}{2}+\dfrac{x}{3}+\dfrac{x}{4}=10$

or $\dfrac{6x}{12}+\dfrac{4x}{12}+\dfrac{3x}{12}=10$

whence $13x=120$

and $x=9\dfrac{3}{13}$

And $9\frac{3}{13}$ divided by 2, 3, and 4, respectively, give $4\frac{8}{13}$, $3\frac{1}{13}$, $2\frac{4}{13}$, the three parts required.

10. Let 10 be divided into 4 parts such, that when they are respectively divided by 2, 3, 4, and 5, the quotients shall be in the same proportion as 6, 7, 8, and 9?

Assume $2 \times 6x$, $3 \times 7x$, $4 \times 8x$, $5 \times 9x$ for the 4 parts; (these being divided by 2, 3, 4, and 5, produce quotients in the given proportions).

$$\begin{aligned} &\text{Then} \ldots\ldots\ldots\ldots & 12x + 21x + 32x + 45x &= 10 \\ &\text{or,} \ldots\ldots\ldots\ldots & 110x &= 10 \\ &\text{and,} \ldots\ldots\ldots\ldots & x &= \frac{1}{11} \end{aligned}$$

Therefore

$$\left.\begin{aligned} \frac{1}{11} \times 12 &= 1\frac{1}{11} \\ \frac{1}{11} \times 21 &= 1\frac{10}{11} \\ \frac{1}{11} \times 32 &= 2\frac{10}{11} \\ \frac{1}{11} \times 45 &= 4\frac{1}{11} \end{aligned}\right\} \text{the 4 parts required.}$$

11. There is a number consisting of two digits, and if 72 be subtracted from it, the digits will be inverted. What is the number?

The answer is found from the following property, namely; The difference of a number consisting of two digits, and the number when those digits are inverted, is 9 *times the difference of the digits:* Thus, if 35 be the number, then the difference of 35, $(3 \times 10 + 5)$ and 53, $(5 \times 10 + 3)$ is 18, or 2 (the difference of 3 and 5) multiplied by 9.

Generally. If a and b are the digits, and $10a + b$ the number, then $10b + a$ is the number when the digits a and b change places: now subtracting the latter from the former, we have

$$10a + b - 10b - a = 9a - 9b = (a - b) \times 9.$$

To apply this to the question, we have only to divide 72 by 9, and the quotient 8 is the difference of the digits; therefore 1 and 9 must be the digits; and 91 the number. For $91 - 72 = 19$.

Corol. Hence the difference between any number, and the number made by its digits in a contrary order, is always divisible by 9.

12. If four agents A, B, C, D, can produce the effects a, b, c, d, in the times f, g, h, k, respectively; in what time would they jointly produce the effect m?

time effect time effect

As $g : b :: f : \frac{bf}{g}$, what B can produce in the time f.

$h : c :: f : \frac{cf}{h}$, what C can produce in the time f.

$k : d :: f : \frac{df}{k}$, what D can produce in the time f.

a, what A can produce in the time f.

Therefore $a+\frac{bf}{g}+\frac{cf}{h}+\frac{df}{k}$ is what they all can produce in the time f, acting together.

Hence, As $a+\frac{bf}{g}+\frac{cf}{h}+\frac{df}{k} : f :: m :$ time required:

or, dividing the two first terms of the proportion by f (92)

$\frac{a}{f}+\frac{b}{g}+\frac{c}{h}+\frac{d}{k} : 1 :: m : m$ divided by $\frac{a}{f}+\frac{b}{g}+\frac{c}{h}+\frac{d}{k}$, the *Ans.*

Corol. Hence, whatever be the number of agents, the required time will be m divided by the sum of the quotients $\frac{a}{f}+\frac{b}{g}+$ &c. For example: Suppose A can dig 50 yards of a trench in 8 days; B, 80 yards in 12 days; and C, 90 yards in 16 days; in what time would they finish 200 yards if they work together?

Here $a=50, f=8$; $b=80$, $g=12$; $c=90$, $h=16$; and $m=200$:

Then $\frac{a}{f}+\frac{b}{g}+\frac{c}{h}=\frac{50}{8}+\frac{80}{12}+\frac{90}{16}=18\frac{13}{24}$; and 200 divided by $18\frac{13}{24}$ gives $10\frac{70}{89}$ days, the *answer*.

13. Suppose the effect m can be produced by the three agents A, B, and C together in the time a, by B, C, and D together in the time b, by C, D, and A in the time c, and by A, B, and D in the time d. Required the time in which each would produce it alone?

Let x, y, z, and v, denote the respective times:

$$\begin{array}{lllll} & \text{time} & \text{effect} & \text{time} & \text{effect} \\ \text{As} & x & : m & :: a & : \frac{ma}{x}, \end{array}$$ what A can produce in the time a.

$$y : m :: a : \frac{ma}{y},$$ what B can produce in the time a.

$$z : m :: a : \frac{ma}{z},$$ what C can produce in the time a.

Now, by the question, the sum of those effects is the effect m,

that is $\frac{ma}{x} + \frac{ma}{y} + \frac{ma}{z} = m$,

or (dividing by m) $\frac{a}{x} + \frac{a}{y} + \frac{a}{z} = 1$

whence $ayz + axz + axy = xyz$.

Again,

$$y : m :: b : \frac{mb}{y},$$ what B can produce in the time b.

$$z : m :: b : \frac{mb}{z},$$ what C can produce in the time b.

$$v : m :: b : \frac{mb}{v},$$ what D can produce in the time b.

And, by the quest. these effects together are equal to the effect m.

viz. $\frac{mb}{y} + \frac{mb}{z} + \frac{mb}{v} = m$; whence $bzv + byv + byz = zyv$.

And proceeding in the same manner with the times c and d,

we get $cvx + czx + czv = zvx$, and $dyv + dxv + dxy = xyv$.

Whence these four equations,

namely,
$$\begin{aligned} ayz + axz + axy &= xyz \\ bzv + byv + byz &= zyv \\ cvx + czx + czv &= zvx \\ dyv + dxv + dxy &= xyv \end{aligned}$$

From the first equation, $ayz + axz - xyz = -axy$

whence $z = \frac{-axy}{ay + ax - xy}$.

By the second equation, $bzv + bzy - zyv = -byv$

whence $z = \frac{-byv}{bv + by - yv}$.

Therefore $\frac{-axy}{ay + ax - xy} = \frac{-byv}{bv + by - yv}$ (the two values of z)

which reduced gives $axv - abx = bvx - abv$.

Again, by the third equation, $czx + czv - zvx = -cxv$

$$\text{whence } z = \frac{-cxv}{cx + cv - vx}$$

$$\text{And } \frac{-byv}{bv + by - yv} = \frac{-cxv}{cx + cv - vx} \text{ (two values of } z \text{ made equal)}$$

This reduced gives $byx - bcy = cxy - bcx$

We now have three equations involving three unknown quantities,

$$\begin{aligned} \text{viz. } axv - abx &= bvx - abv \\ byx - bcy &= cxy - bcx \\ dyv + dxv + dxy &= xyv. \end{aligned}$$

By the first, $axv + abv - bvx = abx$,

$$\text{whence } v = \frac{abx}{ax + ab - bx}.$$

From the third equation, $dyv + dxv - xyv = -dxy$,

$$\text{whence } v = \frac{-dxy}{dy + dx - xy}:$$

$$\text{Therefore } \frac{abx}{ax + ab - bx} = \frac{-dxy}{dy + dx - xy}$$

which reduced becomes $2abdy + abdx - abxy = bdxy - adxy$.

Now v being exterminated, the equations are reduced to the two following, involving only x and y,

$$\begin{aligned} \text{viz. } 2abdy + abdx - abxy &= bdxy - adxy. \\ bxy - bcy &= cxy - bcx. \end{aligned}$$

By the first, $2abdy - abxy - bdxy + adxy = -abdx$

$$\text{whence } y = \frac{-abdx}{2abd - abx - bdx + adx}.$$

From the second, $bxy - bcy - cxy = -bcx$,

$$\text{which gives } y = \frac{-bcx}{bx - bc - cx}:$$

$$\text{Therefore, } \frac{-bcx}{bx - bc - cx} = \frac{-abdx}{2abd - abx - bdx + adx} \text{ (the two values of } y.)$$

$$\text{This reduced gives } x = \frac{3abcd}{abc + bcd + bad - 2adc}.$$

Now the values of y, z, and v, are easily discovered: For $3abcd$ will evidently be a common numerator: And since b is not found in the negative part of the denominator in the expression for x, it follows that $-2abd$, $-2abc$, and $-2bcd$, are the other negative quantities, where c, d, and a, are respectively excluded.

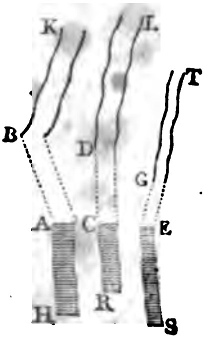
$$\text{Therefore } y = \frac{3abcd}{abc + bcd + acd - 2abd}.$$

$$z = \frac{3abcd}{bcd + acd + abd - 2abc}.$$

$$v = \frac{3abcd}{acd + abd + abc - 2bcd}.$$

A single example proposed in numbers and wrought arithmetically, is less tedious than the preceding operation. But the algebraic method has the advantage of giving general formulæ or expressions for all questions of the kind.

14. Suppose *B* battalions of troops, of equal strength, are in three columns HA, RC, SE, and that they have to pass through the roads or defiles BK, DL, GT, whose breadths admit of different fronts. Let the times of marching from A to K, from C to L, and from E to T, be denoted by *a*, *b*, and *c*, respectively; also put *r*, *s*, and *t* for the respective times in which a battalion can march its own length in BK, DL, and GT. Now it is required to determine the number of battalions of which each column should be composed in order to enable their rears to quit the defiles at K, L, and T, at one and the same time, or that the whole march through the defiles may be made in the least time possible?



Let x, y, and z denote the number of battalions in the columns HA, RC, and SE, respectively.

Then rx, sy, and tz will be the times in which they can march their own lengths in the respective defiles.

$$\text{And } \left.\begin{array}{l} a + rx \\ b + sy \\ c + tz \end{array}\right\} \text{times of marching from} \left\{\begin{array}{l} \text{H to K} \\ \text{R to L} \\ \text{S to T} \end{array}\right.$$

Now, by the question, those times must be equal:

$$\text{or, } a + rx = b + sy$$

$$a + rx = c + tz$$

$$\text{From the first equation } \frac{a - b + rx}{s} = y.$$

But $z = B - x - y$ (because $x + y + z = B$) which put for z in the second equation, gives $a + rx = c + Bt - tx - ty$,

$$\text{whence } \frac{a - c - Bt + rx + tx}{-t} = y:$$

$$\text{Therefore } \frac{a - b + rx}{s} = \frac{a - c - Bt + rx + tx}{-t},$$

$$\text{which reduced gives } x = \frac{Bts + sc + bt - ta - sa}{tr + sr + st}.$$

And repeating the operation for y and z, we have

$$y = \frac{Btr + ta + rc - br - bt}{tr + sr + st}.$$

$$z = \frac{Bst + sa + rb - cr - cs}{tr + sr + st}.$$

Examp. Suppose the number of battalions to be $20 = B$.

And let BK = 2 miles = 4224 paces of $2\frac{1}{2}$ feet each, and the rate of marching 70 paces *per minute.*

DL = $2\frac{1}{4}$ miles = 4752 paces, ——— rate 65 p. *per min.*
GT = 1 mile = 2112 paces, ——— rate 50 p. *per min.*

$$\left.\begin{array}{l}\text{AB} = 1 \text{ mile} = 2112 \text{ p.}\\ \text{CD} = \frac{3}{4} \text{ mile} = 1584 \text{ p.}\\ \text{EG} = \frac{1}{2} \text{ mile} = 1056 \text{ p.}\end{array}\right\} \text{rate of marching 80 p. } \textit{per min.}$$

Paces
205 depth or extent of a battal. in the defile BK
270 .... in DL
350 .... in GT.

Then

$$\frac{4224}{70} = 60.34 \text{ } \textit{min.} \text{ time of marching from B to K.}$$

$$\frac{2112}{80} = \underline{26.4} \text{ } \textit{min.} \text{ time of marching from A to B.}$$

$$\underline{86.74} \text{ } \textit{min.} = a, \text{ time of marching from A to K.}$$

$$\frac{4752}{65} = 73.11 \text{ } \textit{min.} \text{ time of marching from D to L}$$

$$\frac{1584}{80} = \underline{19.8} \text{ } \textit{min.} \text{ time of marching from C to D}$$

$$\underline{92.91} \text{ } \textit{min.} = b, \text{ time of marching from C to L.}$$

$$\frac{2112}{50} = 42.24 \text{ } \textit{min.} \text{ time of marching from G to T}$$

$$\frac{1056}{80} = \underline{13.2} \text{ } \textit{min.} \text{ time of marching from E to G}$$

$$\underline{55.44} \text{ } \textit{min.} = c, \text{ time of marching from E to T.}$$

$\frac{205}{70} = 2.93\ min. = r.$

$\frac{270}{65} = 4.154\ min. = s$

$\frac{350}{50} = 7.0\ min. = t.$

Those values being substituted in the foregoing expressions, give $x = 8$, $y = 4$, and $z = 8$, the nearest integers. Therefore the columns HA, RC, must each consist of 8 battalions, and RC of 4.

In this example, the three columns are supposed to begin their march at the same time: but should it be found necessary to delay the movement of either column, the numeral value of the corresponding letter must be varied accordingly, and a new division of the battalions take place. Thus suppose the troops at A and C are to begin their march 25 minutes before those which pass the defile GT,

Then c will be $55.44 + 25 = 80.44$ *min.* and the resulting values of x, y, and z, are 10, 5, and 5 (the nearest integers) for the number of battalions in the columns AH, RC, and SE, so that the whole body may clear the defiles in the least time possible, *in that case.*

Should the value of either expression be less than $\frac{1}{2}$, the whole body will pass in two columns only: Thus suppose the rate of marching in DL, the middle defile, is only 45 paces *per minute*:

$$\text{Then } \frac{4752}{45} = 105.6, \text{ and } 105.6 + 19.8 = 125.4\ min. = b.$$

$$\text{And } \frac{270}{45} = 6\ min = s. \text{ Whence } y = \tfrac{1}{10} \text{ nearly;}$$

And retaining a, c, r, and t, as in the first example, x and z will be found 12 and 8, (the nearest whole numbers) respectively, for the number of battalions in the columns HA, and SE.

But when it is proposed to make the division for two roads or defiles only, the expressions become much more simple; for in that case we have but two equations,

$$\text{namely, } a + rx = b + sy;$$
$$\text{and } x + y = B.$$

$$\text{Whence } x = \frac{Bs + b - a}{r + s}, \text{ and } y = \frac{Br + a - b}{r + s}.$$

Now suppose the 20 battalions are to march through BK, and DL only; and let $a = 86.74$, $b = 92.91$, $r = 2.93$, and $s = 4.154$, as before;

Then $x = \frac{89.25}{7.084} = 12.6$, and $y = \frac{52.43}{7.084} = 7.4$

or 13, and 7 battalions in the columns HA, and RC, respectively.

15. In drawing up a certain number of men into a square column, it was found that 21 men were left; but when the side of the square was increased by 1 man, then 34 men were wanting to complete the square. Required the number of men?

Let x be the number of men in the side of the first square;

Then $x^2 + 21$ is the whole number of men:

And $(x+1)^2 - 34$, or $x^2 + 2x + 1 - 34$, is also the number:

Therefore $x^2 + 21 = x^2 + 2x - 33$

or $21 = 2x - 33$

Whence $x = 27$; therefore $27^2 + 21 = 750$, the *answer.*

16. To find 3 numbers in Harmonical Proportion, when the difference of the first and second is denoted by a, and that of the second and third by b.

If 3 numbers are in musical proportion, the first will be to the third, as the difference between the first and second, is to the difference of the second and third.

Let the first number be x;

Then the second will be $x + a$;

And the third............ $x + a + b$.

Whence, as $x : x + a + b :: a : b$

therefore $ax + a^2 + ab = bx$

and $x = \frac{a^2 + ab}{b - a}$.

Let $a = 2$, $b = 3$. Then $\frac{a^2 + ab}{b - a} = 10$, the first number.

$10 + 2 = 12$ the second.

$10 + 2 + 3 = 15$ the third.

This Harmonic Proportion relates to the lengths of Musical Cords. Thus, if 6 strings of equal thickness and tension, are made to sound or vibrate together, the greatest harmony they can produce, will be when their lengths are in the proportion of $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$: Whence those fractions are said to be in Musical Proportion.

Since the denominators 1, 2, 3, 4, 5, 6, are in Arithmetical proportion, it follows, that numbers in harmonic proportion are the reciprocals of numbers in arithmetical proportion, (and *vice versa*). For $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, are the reciprocals of 1, 2, 3, 4, 5, 6.

If the fractions are reduced to a common denominator, the numerators will be 60, 30, 20, 15, 12, 10; which are 6 numbers in harmonic proportion.

17. What is the least number of weights, and the weight of each, that will weigh any number of pounds from 1lb. to an Hundred weight?

It is evident that one of the weights must be 1lb:

Now let x denote the next greater weight: then in order to weigh 2lb. $x-1$ must be equal to 2,

viz. $x-1=2$, or $x=3$:

And since $3+1=4$, it follows that 1, 2, 3, and 4lb. may be weighed with 1lb. and a 3lb.

Again, put x for the *greatest weight next greater* than 3: Then, to weigh 5lb. with the weights 1, 3, and x, the value of $x-3-1$ must not exceed 5;

therefore making $x-3-1=5$, gives $x=9$.

And $9+3+1=13$; consequently any number of pounds up to 13 may be weighed with the weights 1, 3, and 9.

$$\begin{array}{llll}
\text{For } 9+3+1 & =13 & 9 & =8+1 \\
9+3 & =12 & 9+1 & =7+3 \\
9+3 & =11+1 & 9 & =6+3 \\
9+1 & =10 & 9 & =5+3+1 \\
9 & =9 & &
\end{array}$$

And if x denote the weight next greater than 9; then $x-9-3-1=14$; and $x=27$: Hence it appears, that the least number of weights, in all cases, will be the geometric series 1, 3, 9, 27, 81, 243, &c. The first 5 however, are sufficient in the present example: for $1+3+9+27+81=121$lb. that may be weighed with those 5 weights.

OF QUADRATIC EQUATIONS.

129. It has already been observed (74) that when the highest power of the unknown quantity in an equation is of two dimensions, the equation is called a *Quadratic.*

If the equation involves the square only, it is a simple quadratic; as $x^2 = bc$, where the value of x is $= \sqrt{bc}$.

But when one term contains the square, and another its root, the equation is an adfected or affected one: These are all reducible to the three following forms:

$$x^2 + ax = b.$$
$$x^2 - ax = c.$$
$$x^2 - ax = -d.$$

The method of resolving these equations is easily deduced from the square of a binomial, thus:

Let $x + r$ be the binomial, then its square is $x^2 + 2rx + r^2$, in which we are to remark, that half $2r$ the coefficient of x in the middle term, is r the root of r^2 the third term. Therefore the third term of the square of which $x^2 + ax$ are the two first terms, will be $\frac{1}{4}a^2$ the square of half the coefficient a; the whole square being $x^2 + ax + \frac{1}{2}a^2$, and its root $x + \frac{1}{2}a$. This is called *completing the square.*

To find the value of x in the equation $x^2 + ax = b$: add $\frac{1}{4}a^2$ to each side of the equation, and we have

$$x^2 + ax + \tfrac{1}{4}a^2 = b + \tfrac{1}{4}a^2 \text{ (75. Ax. 1.)}$$

And taking the root of each side, $x + \frac{1}{2}a = \sqrt{(b + \frac{1}{4}a^2)}$ (*Ax.* 7.)

whence $x = \sqrt{(b + \frac{1}{4}a^2)} - \frac{1}{2}a$.

But (104) the square root of $b + \frac{1}{4}a^2$ is also denoted by $-\sqrt{(b + \frac{1}{4}a^2)}$, therefore $x = -\sqrt{(b + \frac{1}{4}a^2)} - \frac{1}{2}a$, which is the negative value of x; the former being the affirmative one.

The value of x in the second equation is found in the same manner: for by adding $\frac{1}{4}a^2$ (the square of half the coefficient a) to each side of the equation, we get

$$x^2 - ax + \tfrac{1}{4}a^2 = c + \tfrac{1}{4}a^2$$

and extracting the roots, ...$x - \frac{1}{2}a = \sqrt{(c + \frac{1}{4}a^2)}$

whence $x = \frac{1}{2}a + \sqrt{(c + \frac{1}{4}a^2)}$

which is the affirmative root.

But in this case, $\frac{1}{2}a - x$ is also the square root of $x^2 - ax + \frac{1}{4}a^2$, for $(\frac{1}{2}a - x)^2 = x^2 - ax + \frac{1}{4}a^2$,

therefore $\frac{1}{2}a - x = \sqrt{(c + \frac{1}{4}a^2)}$

which gives $x = \frac{1}{2}a - \sqrt{(c + \frac{1}{4}a^2)}$, the negative root.

This ambiguity is usually denoted by means of the double or uncertain sign $\pm$, thus $x = \frac{1}{2}a \pm \sqrt{(c + \frac{1}{4}a^2)}$.

By completing the square, and extracting the roots in the third form $x^2 - ax = -d$,

we get $x = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - d)}$, where both roots or expressions will be affirmative. For since $\sqrt{\frac{1}{4}a^2}$ is $= \frac{1}{2}a$, $\sqrt{(\frac{1}{4}a^2 - d)}$ must be less than $\frac{1}{2}a$; therefore $\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - d)}$ will be affirmative.

If d be greater than $\frac{1}{4}a^2$, then $\sqrt{(\frac{1}{4}a^2 - d)}$ is an imaginary quantity whose root cannot be assigned; in which case the roots, or values of x are both impossible.

Respecting the two affirmative roots $\frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^4 - d)}$ that result from this third form, the nature of the problem will determine which is to be taken: both values however, frequently answer the conditions of the question.

130. *Other Examples.*

1. Given $x^2 + 8x = 209$. To find x.

The square of $\frac{1}{2}$ the coefficient 8 is 16,

Therefore $x^2 + 8x + 16 = 209 + 16$,

or $x^2 + 8x + 16 = 225$.

And taking the square roots, $x + 4 = \sqrt{225} = 15$,
whence $x = 15 - 4 = 11$, the *Answer*.

2. Given $x^2 - 6x = 72$. To find x.

The square of half 6 is 9;

Whence $x^2 - 6x + 9 = 72 + 9 = 81$,

And by evolution, $x - 3 = \sqrt{81} = 9$.

Therefore $x = 3 \pm 9 = 12$ and -6, the positive, and negative roots, or values of x.

3. Given $x^2 - 12x = -35$. To find x.

Completing the square gives $x^2 - 12x + 36 = 36 - 35 = 1$.

And by evolution we get $x - 6 = 1$,

whence $x = 6 \pm 1 = 7$ and 5 the two roots:

And both answer the conditions of the question:

For $7^2 - 12 \times 7 = -35$
And $5^2 - 12 \times 5 = -35$.

4. Given $3x^2 + 21x = 180$. To find x.

In this, and all other examples in which the square of the unknown quantity is affected with a coefficient, it is evident the whole equation must be divided by that coefficient before the square can be completed.

Now dividing by 3 gives $x^2 + 7x = 60$

And completing the square, $x^2 + 7x + 12\frac{1}{4} = 60 + 12\frac{1}{4} = 72\frac{1}{4}$.

Whence, by evolution, $x + 3\frac{1}{2} = \sqrt{72\frac{1}{4}} = 8\frac{1}{2}$
therefore $x = 8\frac{1}{2} - 3\frac{1}{2} = 5$, the *Answer*.

5. Given $x^2 - x = -\frac{2}{9}$. To find the value of x.

Here 1 is the coefficient of the second term x. Therefore the square being compleated, we have $x^2 - x + \frac{1}{4} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$;

And taking the roots,......$x - \frac{1}{2} = \sqrt{\frac{1}{36}} = \frac{1}{6}$,
whence.........$x = \frac{1}{2} \pm \frac{1}{6} = \frac{2}{3}$ and $\frac{1}{3}$ the two values of x; both of which answer the question.

6. Given $ax^2 - x = b$. To find x.

The whole divided by a gives.......... $x^2 - \frac{1}{a}x = \frac{b}{a}$,

And completing the square...... $x^2 - \frac{1}{a}x + \frac{1}{4a^2} = \frac{b}{a} + \frac{1}{4a^2}$;

Whence, by evolution.................. $x - \frac{1}{2a} = \sqrt{\left(\frac{b}{a} + \frac{1}{4a^2}\right)}$.

Therefore $x = \frac{1}{2a} \pm \sqrt{\left(\frac{b}{a} + \frac{1}{4a^2}\right)}$.

7. Given $ax^2 + bx^2 + dx - cx = m$. To find x.

Dividing by $a + b$ the coefficient of x^2, we have

$$x^2 + \frac{d-c}{a+b}x = \frac{m}{a+b}$$

And completing the square, $x^2 + \frac{d-c}{a+b}x + \left(\frac{d-c}{2a+2b}\right)^2 = \frac{m}{a+b} + \left(\frac{d-c}{2a+2b}\right)^2$

By evolution,........................ $x + \frac{d-c}{2a+2b} = \left(\frac{m}{a+b} + \left(\frac{d-c}{2a+2b}\right)^2\right)^{\frac{1}{2}}$

whence $x = \left(\frac{m}{a+b} + \left(\frac{d-c}{2a+2b}\right)^2\right)^{\frac{1}{2}} - \frac{d-c}{2a+2b}$.

8. Given $x + \sqrt{x} = a$. To find x.

By transposition......................... $a - x = \sqrt{x}$

And squaring both sides....... $a^2 - 2ax + x^2 = x$

Whence, by transposition,... $x^2 - 2ax - x = -a^2$

or $x^2 - (2a + 1)x = -a^2$

And completing the square $x^2 - (2a + 1)x + (a + \frac{1}{2})^2 = (a + \frac{1}{2})^2 - a^2$

or $x^2 - (2a + 1)x + (a + \frac{1}{2})^2 = a + \frac{1}{4}$

And taking the roots...................... $x - (a + \frac{1}{2}) = \pm \sqrt{(a + \frac{1}{4})}$

whence $x = a + \frac{1}{2} \pm \sqrt{(a + \frac{1}{4})}$.

Let $a = 15\frac{3}{4}$. Then $x = 16\frac{1}{4} \pm 4 = 12\frac{1}{4}$ and $20\frac{1}{4}$, the two values of x; but it is only the first which answers the conditions of the equation. The other value $20\frac{1}{4}$ is what would result, supposing $x - \sqrt{x} = 15\frac{3}{4}$ (a); for in that case, $-\sqrt{x} = a - x$, whence, by squaring both sides, we get

$x = a^2 - 2ax + x^2$ as before, because the square of $-\sqrt{x}$ is the same as that of $\sqrt{x}$.

Sometimes the process of resolving an equation may be abridged by making use of a substitution, as in the two next examples.

9. Given $\sqrt{(a+x)} - b(a+x)^{\frac{1}{4}} = m$. To find x.

Let $z^4 = a + x$.

Then……………………$z = (a+x)^{\frac{1}{4}}$

And……………………$z^2 = \sqrt{(a+x)}$.

Therefore………………$z^2 - bz = m$;

Whence, by completing the square, and extracting the roots,

we get $z = \frac{1}{2}b \pm \sqrt{(m + \frac{1}{4}b^2)}$.

Consequently $z^4 = [\frac{1}{2}b \pm \sqrt{(m + \frac{1}{4}b^2)}]^4$;

or $a + x = [\frac{1}{2}b \pm \sqrt{(m + \frac{1}{4}b^2)}]^4$;

Therefore $x = [\frac{1}{2}b \pm \sqrt{(m + \frac{1}{4}b^2)}]^4 - a$.

10. Given $x^2 + xy = 918$.

$xy - 3y^2 = 42$. To find x and y.

Put $zy = x$:

Then $z^2y^2 + zy^2 = 918$: whence $y^2 = \frac{918}{z^2 + z}$.

And $zy^2 - 3y^2 = 42$: whence $y^2 = \frac{42}{z - 3}$:

Therefore $\frac{42}{z - 3} = \frac{918}{z^2 + z}$:

And, by reduction, $42z^2 + 42z = 918z - 2754$:

whence $z^2 - 20_7 z = -65\frac{4}{7}$:

Now by completing the square, and extracting the roots,

we get $z = 10\frac{3}{7} \pm 6\frac{4}{7} = 17$, and $3\frac{6}{7}$. These values being substituted for z in the equation $y^2 = \frac{42}{z - 3}$, give $y = \sqrt{3}$, and $y = 7$; And the corresponding values of x will be $17\sqrt{3}$, and 27.

Therefore $\left.\begin{array}{l} x = 27 \\ y = 7 \end{array}\right\}$ the rational values: And $\left.\begin{array}{l} x = 17\sqrt{3} \\ y = \sqrt{3} \end{array}\right\}$ the irrational.

11. Given $x^6 - 4x^3 = 621$. To find x.

Since x^3 is the square root of x^6, this equation is solved after the manner of a quadratic: thus,

Add 4, the square of half the coefficient 4, to each side of the equation, and we have

$$x^6 - 4x^3 + 4 = 645$$

And extracting the square roots, $x^3 - 2 = 25$

whence $x^3 = 27$; and $x = 3$.

In general; any equation of this form $x^{2n} - ax^n = b$, (where x is the unknown quantity, and the indices $2n$, and n, are one double the other,) is resolved in the same manner:

For by adding $\frac{1}{4}a^2$ to each side of the equation,

we have $x^{2n} - ax^n + \frac{1}{4}a^2 = b + \frac{1}{4}a^2$

Now $x^n - \frac{1}{2}a$ is the square root of $x^{2n} - ax^n + \frac{1}{4}a^2$,

Therefore $x^n - \frac{1}{2}a = \sqrt{(b + \frac{1}{4}a^2)}$

whence $x^n = \frac{1}{2}a \pm \sqrt{(b + \frac{1}{4}a^2)}$

And $x = [\frac{1}{2}a \pm \sqrt{(b + \frac{1}{4}a^2)}]^{\frac{1}{n}}$.

131. QUESTIONS PRODUCING QUADRATIC EQUATIONS.

1. To divide 40 into two such parts, that the sum of their squares shall be 818?

Let one of the parts be................x

Then the other will be ....$40 - x$

By the question............$(40 - x)^2 + x^2 = 818$

or $1600 - 80x + x^2 + x^2 = 818$

that is $2x^2 - 80x = 818 - 1600 = -782$

or $x^2 - 40x = -391$

And completing the square $x^2 - 40x + 400 = 9$

By evolution $x - 20 = 3$

whence $x = 20 \pm 3 = 23$ and 17, the two parts required.

2. To divide 11 into two such parts, that the sum of their cubes may be 407?

Let one part be................................x

Then the other must be............$11 - x$

And, by the question.......$(11 - x)^3 + x^3 = 407$

or....$1331 - 363x + 33x^2 - x^3 + x^3 = 407$

that is.... ...$1331 - 363x + 33x^2 = 407$

or.....................$33x^2 - 363x = -924$

And dividing by 33............................$x^2 - 11x = -28$

And completing the square.............$x^2 - 11x + 30\frac{1}{4} = 2\frac{1}{4}$

whence, by evolution,..........................$x - 5\frac{1}{2} = 1\frac{1}{2}$

therefore $x = 5\frac{1}{2} \pm 1\frac{1}{2}$

$=$ 7 and 4, the parts required.

3. To divide 146 into two such parts, that the difference of their square roots may be 6.

Suppose the least root to be...........x

Then the other must be...............$x + 6$

Whence, by the question, $(x + 6)^2 + x^2 = 146$

that is $2x^2 + 12x + 36 = 146$

or $x^2 + 6x + 18 = 73$

Whence $x^2 + 6x = 55$

And completing the square, $x^2 + 6x + 9 = 55 + 9 = 64$;

By evolution, $x + 3 = 8$

and $x = 5$, one of the roots; whence $5 + 6 = 11$ the other. And the two parts are 5^2 and 11^2.

4. The sum of two numbers being $2s$, and the sum of their 4*th.* powers p, to find the numbers.

Let $\frac{1}{2}$ the difference of the numbers be denoted by x; then $\frac{1}{2}$ their sum being s,

we have $s + x$ for the greater,
and $s - x$ the less. (128. *Ex.* 3.)

Then, by the question, $(s + x)^4 + (s - x)^4 = p$.

Now $(s + x)^4 = s^4 + 4s^3 x + 6s^2 x^2 + 4x^3 s + x^4$
$(s - x)^4 = s^4 - 4s^3 x + 6s^2 x^2 - 4sx^3 + x^4$

sum $2s^4 + 12s^2 x^2 + 2x^4$

or.................................$2x^4 + 12s^2 x^2 + 2s^4 = p$;

And dividing by 2 ....$x^4 + 6s^2 x^2 + s^4 = \frac{1}{2}p$,

or...$x^4 + 6s^2 x^2 = \frac{1}{2}p - s^4$

And completing the square, $x^4 + 6s^2 x^2 + 9s^4 = \frac{1}{2}p + 8s^4$.

Whence, by evolution,......................$x^2 + 3s^2 = \sqrt{(\frac{1}{2}p + 8s^4)}$

Therefore..................$x^2 = \sqrt{(\frac{1}{2}p + 8s^4)} - 3s^2$

and....................$x = (\sqrt{(\frac{1}{2}p + 8s^4)} - 3s^2)^{\frac{1}{2}}$.

Suppose the sum $= 12 = 2s$ (or $s = 6$) and the sum of the biquadrates $= 3026 = p$.

Then, those values substituted for s and p in the expression for x, and we have $x = \sqrt{(109 - 108)} = 1$.

Therefore $6 \pm 1 = 7$ and 5, the two numbers.

5. The sum (s), and sum of the squares (p) of four numbers in arithmetical progression, being given; to find the numbers.

Numbers in arithmetical progression have a common difference (121. Arith.) Therefore, if $2x$ be that difference, and $4a = s$.

Then $a - 3x$, $a - x$, $a + x$, $a + 3x$, will denote the four numbers; for their sum is $4a$ (or s), and common difference $2x$.

Now by the question, $(a-3x)^2+(a-x)^2+(a+x)^2+(a+3x)^2=p$.

$$\begin{aligned}(a-3x)^2 &= a^2-6ax+9x^2\\ (a-x)^2 &= a^2-2ax+x^2\\ (a+x)^2 &= a^2+2ax+x^2\\ (a+3x)^2 &= a^2+6ax+9x^2\end{aligned}$$

$$\text{sum}\quad 4a^2 \qquad +20x^2=p:$$

$$\text{or}\quad 20x^2=p-4a^2=p-\tfrac{1}{4}s^2$$

$$\text{whence}\quad x=\sqrt{\left(\frac{4p-s^2}{80}\right)}$$

Or the whole difference $2x=\sqrt{\frac{4p-s^2}{20}}$.

And in the same manner, if the number of terms be three, their common difference will be found $=\sqrt{\frac{3p-s^2}{6}}$. Also if 5 be the number, the common difference is $\sqrt{\frac{5p-s^2}{50}}$. Hence it appears, that the coefficient of p is the number of terms; and that the denominators 6, 20, 50, &c. resulting from the coefficients of x^2, are each equal to half the number of terms drawn into the sum of the series $1+3+6+10+$ &c. continued to $n-1$ terms, n being the number of terms whose sum is given.

Now the sum of the series $1+3+6+10$, &c. continued to $n-1$ terms, is $n\times\frac{n^2-1}{6}$ (144), which drawn into $\frac{n}{2}$ gives $\frac{n^2}{2}\times\frac{n^2-1}{6}$:

Therefore, if the sum (s), and sum of their squares (p) of any number (n) of terms in arithmetical progression, are given, then $\sqrt{\left(\frac{np-s^2}{\frac{n^2}{2}\times\frac{n^2-1}{6}}\right)}$ or its equal $\frac{2}{n}\sqrt{\frac{3np-3s^2}{n^2-1}}$ is the common difference of the terms.

Examp. Let the number of terms be 7, their sum $=49$, and the sum of their squares $=455$; then, putting those numbers for n, s, and p, respectively, we get 2 for the difference; whence the numbers are 1, 3, 5, 7, 9, 11, and 13.

6. The sum (s) and continued product (p) of 5 numbers in arithmetical progression, being given; to find those numbers.

Let $5a = s$; and put $2x$ for the common difference.

Then $a - 4x$, $a - 2x$, a, $a + 2x$, $a + 4x$, will denote the 5 numbers.

Therefore, by the Quest. $(a - 4x)(a - 2x)(a)(a + 2x)(a + 4x) = p$;

or $64ax^4 - 20a^3x^2 + a^5 = p$:

Whence, by division, $x^4 - \frac{5}{16}a^2x^2 + \frac{a^4}{64} = \frac{p}{64a}$

or $x^4 - \frac{5}{16}a^2x^2 = \frac{p - a^5}{64a}$.

Now, by completing the square and extracting the roots,

we get $x^2 = \frac{5}{32}a^2 \pm \sqrt{\frac{16p + 9a^5}{32^2 a}}$.

If $s = 25$, and $p = 945$; then a being $= 5$; we shall have $x^2 = 1$; and $x = 1$, therefore $2x = 2$ the common difference of the numbers or terms; whence the 5 numbers are found to be 1, 3, 5, 7, and 9.

7. The sum (s) and product (p) of any two numbers being given; to find the sum of their squares, cubes, biquadrates, &c.

Let the two numbers be .... x and y

Then, by the question, ... $x + y = s$

and .... $xy = p$

The first equation squared gives .... $x^2 + 2xy + y^2 = s^2$

whence .... $x^2 + y^2 = s^2 - 2xy$

But $2xy = 2p$; therefore $x^2 + y^2 = s^2 - 2p$, the sum of the squares.

Now multiply the last equation by the first,

and we have .... $(x^2 + y^2)(x + y) = (s^2 - 2p)s$

or .... $x^3 + xy(x + y) + y^3 = s^3 - 2sp$.

But $xy(x + y) = sp$, which substituted in the last equation

gives $x^3 + sp + y^3 = s^3 - 2sp$

or $x^3 + y^3 = s^3 - 3sp$ the sum of the cubes.

Again, let this last equation be multiplied by the first,

Then $(x^3 + y^3)(x + y) = (s^3 - 3sp)s$

or $x^4 + xy(x^2 + y^2) + y^4 = s^4 - 3s^2p$

But $xy(x^2+y^2)=p(s^2-2p)$, therefore, by substitution,

$$x^4+p(s^2-2p)+y^4=s^4-3s^2p$$

or $x^4+y^4=s^4-3s^2p-p(s^2-2p)=s^4-4s^2p+2p^2$ the sum of the *4th. powers.*

And the sum of the *5th.* powers will be

$$(s^4-4s^2p+2p^2)s-(s^3-3sp)p:$$

That is, the sum of the next superior powers is constantly obtained by multiplying the sum of the powers last found by s, and subtracting from that product the sum of the next preceding ones multiplied by p.

And the sum of the *n*th. powers will be

$s^n-ns^{n-2}p+n\times\frac{n-3}{2}\times s^{n-4}p^2-n\times\frac{n-4}{2}\times\frac{n-5}{3}\times s^{n-6}p^3$ $+n\times\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}\times s^{n-8}p^4$, &c. Where it is evident the series, or expression for the sum of the powers, will terminate when the least index of s becomes $=0$.

8. Given the sum (s), and sum of the squares (p) of four numbers in geometrical progression; to find those numbers.

Let the first number be denoted by $\frac{y^2}{x}$, and the common ratio by $\frac{x}{y}$;

Then the 4 numbers will be $\frac{y^2}{x}$, y, x, $\frac{x^2}{y}$, (142. Arith.)

By the question................ $\frac{y^2}{x}+y+x+\frac{x^2}{y}=s$

and.................... $\left(\frac{y^2}{x}\right)^2+y^2+x^2+\left(\frac{x^2}{y}\right)^2=p$.

Put $x+y=z$, and $xy=v$. Then $\frac{y^2}{x}+\frac{x^2}{y}=s-z$,

And.................................. $\frac{y^2}{x}\times\frac{x^2}{y}=xy=v$ (144 Arith.)

But, by the preceding question, if the sum $x+y=z$, and the product $xy=v$; then $x^2+y^2=z^2-2v$.

And for the like reason, if the sum $\frac{y^2}{x}+\frac{x^2}{y}=s-z$, and the product $\frac{y^2}{x}\times\frac{x^2}{y}=v$,

then........................ $\left(\frac{y^2}{x}\right)^2+\left(\frac{x^2}{y}\right)^2=(s-z)^2-2v$.

Consequently the sum of the four squares will be

$$x^2+y^2+\left(\frac{y^2}{x}\right)^2+\left(\frac{x^2}{y}\right)^2=z^2-2v+(s-z)^2-2v=z^2+(s-z)^2-4v=p.$$

And since.............................. $\frac{y^2}{x}+\frac{x^2}{y}=s-z,$

we have, by reduction,........ $y^3+x^3=xy\,(s-z)$

Or, putting v for its equal xy, gives.... $y^3+x^3=v\,(s-z)=vs-vz$

But, by the preceding quest.......... $y^3+x^3=z^3-3vz$

Therefore, by equality, ... $z^3-3vz=vs-vz$

whence.................. $v=\frac{z^3}{2z+s}$

this value of v substituted in the equation $z^2+(s-z)^2-4v=p$

gives............ $z^2+(s-z)^2=\frac{4z^3}{2z+s}=p$

which reduced becomes...... $z^2+\frac{p}{s}z=\frac{s^2-p}{2}$

whence........ $z=\sqrt{\left(\frac{s^2-p}{2}+\frac{p^2}{4s^2}\right)}-\frac{p}{2s}.$

Examp. Let $s=80$, and $p=3280$. Then $z=24$, which put for z in the equation $v=\frac{z^3}{2z+s}$, gives $v=108$. Therefore $x+y=24$, and $xy=108$; whence $x=6$, and $y=18$: and the 4 numbers are 2, 6, 18, 54.

9. To find two numbers such, that their sum, product, and sum of their squares shall, if possible, be equal to each other.

Let the two numbers be represented by x and y.

Then, by the question,... .... $x+y=xy$

and.......... $x^2+y^2=xy$

Now, if $2xy$ be added to each side of this last equation,

we have............ $x^2+2xy+y^2=3xy$

And extracting the square root... $x+y=\sqrt{3xy}$

But $x+y=xy$, therefore by equality, $\sqrt{3xy}=xy$

whence......... $3xy=x^2y^2$

Consequently, by division.... .... $3=xy$

But $x+y=xy$, therefore............... $x+y=3$

And $x+y=x^2+y^2$, whence.......... $x^2+y^2=3$

Now $x+y$ being $=3$, we have $x=3-y$

Therefore......... .... $(3-y)^2+y^2=3$

From this equation...$y-1\frac{1}{2}=\sqrt{-\frac{3}{4}}$, which being an impossible quantity, it follows that no two numbers can be found to answer the conditions of the question.

10. A regiment of Foot was ordered to send 216 men on Garrison duty, each Company to furnish a like number; but before the detachment marched, three of the Companies were sent on another service, when it was found that each Company which remained was obliged to furnish 12 additional men in order to make up the complement 216. Hence the number of Companies composing the regiment is required?

Let the number of companies in the regiment be... x

Then the number of men which each would have sent, will be $\frac{216}{x}$

But the number of Companies left when 3 were sent away, is $x-3$

And the number of men which each sent on Garrison duty, in that case, is.. $\frac{216}{x-3}$

Therefore the difference must be 12,...... viz. $\frac{216}{x-3} - \frac{216}{x} = 12$

which reduced gives......... $x^2 - 3x = 54$

whence $x = 9$, the *Answer.*

11. The number of men in both fronts of two columns of troops A and B when each consisted of as many ranks as it had men in front, was 84: but when the columns changed ground, or A was drawn up with the front that B had, and B with the front that A had, then the number of ranks in both columns was 91. Required the number of men in each column?

Let x^2 and y^2 denote the men in the two columns, respectively:

Then, by the question, $x + y = 84$.

And $\frac{x^2}{y}$ was the number of ranks in the column x^2 when drawn up with the front y;

And $\frac{y^2}{x}$ the number of ranks in the column y^2 with the front x. Therefore by the quest. $\frac{x^2}{y} + \frac{y^2}{x} = 91$.

Put $z = \frac{x-y}{2}$, and $a = \frac{84}{2} = 42$

Then $x = a + z$, and $y = a - z$ (128 *Ex.* 3.)

And $\frac{(a+z)^2}{a-z} + \frac{(a-z)^2}{a+z} = 91$

$$\text{Or } (a+z)^3 + (a-z)^3 = 91\,(a^2-z^2)$$

Which reduced gives $z^2 = \dfrac{91a^2 - 2a^3}{91 + 6a} = 36$; whence $z = 6$.

Therefore $42 \pm 6 = 48$, and 36, the values of x and y.

And $\left.\begin{array}{l} 48^2 = 2304 \\ 36^2 = 1296 \end{array}\right\}$ The No. of Troops.

OF UNLIMITED OR INDETERMINATE PROBLEMS.

132. IF the independent equations expressing the conditions of a Question are fewer in number than the unknown quantities they involve, the Problem is said to be indeterminate or unlimited (81) because it frequently admits of innumerable answers. But the number of results are generally limited by restricting the values of the unknown quantities to integers. Thus, if $x + y = 5$, then x and y may be any two numbers whatever, whose sum is 5; but 1, 4, 2 and 3 are all their integral values.

133. When the relation of two unknown quantities only are expressed in a simple equation, let the whole equation be divided by the least of the two coefficients, then put the fractional part of the quotient equal to some letter denoting a whole number, positive, or negative, according to the value of the fraction, and a new equation will be obtained; then proceed with the least coefficient as before, and so on, till the last assumed letter has 1 for its coefficient resulting from division; and the expression will come under one of these forms, $\dfrac{z \pm n}{m}$, or $\dfrac{z}{m}$, where the unknown quantity or letter (z) may, in general, be assumed so as to give two extreme integral values in the required answer, whence the others are readily found.

Examples.

1. Given $9x + 13y = 200$: required the values of x and y in whole positive numbers.

By transposition................. $9x=200-13y$

And dividing the whole equation by 9... $x=\frac{200-13y}{9}=22+\frac{2-4y}{9}-y$.

Now x and y are to be whole numbers, therefore $\frac{2-4y}{9}$ must also be a whole number, because the sums or differences of whole numbers are integers.

Also, since y is a whole number, $4y$ must be greater than 2, and consequently $\frac{2-4y}{9}$ the fractional part of the quotient, will be negative;

Therefore, put $\frac{2-4y}{9}=-a$ (a negative whole number)

Then $2-4y=-9a$

Whence $4y=9a+2$

Therefore $y=\frac{9a+2}{4}=2a+\frac{a+2}{4}$

Now to make $\frac{a+2}{4}$ a positive integer, a must be expounded by some term in the series 2, 6, 10, 14, &c., but its least possible value is when $a=2$, the expression in that case being 1; which gives 5 for the least value of y; and the corresponding or greatest value of x is 19.

If a be taken $=6$; then $y=14$, and $x=2$

$$\text{Therefore } \begin{array}{r|l} y=5 & 14. \\ x=19 & 2. \end{array}$$

Which are all the values in positive integers.

The foregoing process is evidently analogous to that of finding the greatest common measure of two numbers in Arithmetic, (40. Arith.) and founded on the same principle, namely, *if a number measures another number, and also a part of that number, it will measure the remaining part.*

2. Given $256x-87y=1$; to find the least possible values of x and y in whole positive numbers.

By transposition ... . $87y=256x-1$

And dividing by the least coefficient 87, gives $y=2x+\frac{82x-1}{87}$

Let $\frac{82x-1}{87}=a$; then $82x-1=87a$

whence $x=\frac{87a+1}{82}=a+\frac{5a+1}{82}$

Next, put $\frac{5a+1}{32}=b$; then $5a+1=32b$

$$\text{whence } a=\frac{32b-1}{5}=16b+\frac{2b-1}{5}:$$

Now assume $\frac{2b-1}{5}=c$; then $2b=5c+1$

$$\text{whence } b=\frac{5c+1}{2}=2c+\frac{c+1}{2}:$$

If $c=1$, then $\frac{c+1}{2}=1$ the least possible integer. Whence $b=3$, and $a=49$; therefore $x=52$, and the corresponding value of y is 153, the required values of x and y.

The other values of x and y are unlimited in number, because c may be any integer so that $c+1$ is divisible by 2.

3. Given $19x-14y=11$; to find the least possible values of x and y in whole numbers.

By transposition.................. $14y=19x-11$

And dividing by 14 gives....... $y=x+\frac{5x-11}{14}$

Put $\frac{5x-11}{14}=a$; then $5x-11=14a$,

$$\text{whence } x=\frac{14a+11}{5}=2a+\frac{4a+1}{5}+2.$$

Next, let $\frac{4a+1}{5}=b$; then $a=b+\frac{b-1}{4}$: now to make this expression the least possible integer (1) the value of b must be 5, in that case $x=19$, and $y=25$, which are two values of the unknown quantities, but not the least; these however, are discovered from the fraction $\frac{4a+1}{5}$ which, when $a=1$, becomes 1, and x $\left(=2a+\frac{4a+1}{5}+2\right)=5$, and the corresponding value of y is 6; the numbers answering the conditions of the question.

4. Given $5x+7y=29$; to find x and y in positive integers.

From the given equation, $x=\frac{29-7y}{5}=5+\frac{4-2y}{5}-y$:

Let $\frac{4-2y}{5}=a$; then $4-2y=5a$

$$\text{whence } y=\frac{4-5a}{2}=2-\frac{a}{2}-2a:$$

Now if $2-\frac{a}{2}-2a$ (or y) is a positive integer, a cannot be any affirmative whole number whatever; therefore making $a=0$, y becomes $=2$, and thence $x=3$; which are all the integral values of x and y in the proposed equation.

5. Let $11x+16y=100$; required the values of x and y in positive integers.

By transposition we get............ $11x=100-16y$

whence...... $x=\frac{100-16y}{11}=9+\frac{1-5y}{11}-y$:

Let $\frac{1-5y}{11}=-a$ ($\frac{1-5y}{11}$ being evidently negative)

Then $1-5y=-11a$, whence $5y=11a+1$,

and $y=\frac{11a+1}{5}=2a+\frac{a+1}{5}$, where the least value of a to make this a whole number, must be 4, which gives $y=8+1=9$:

But from the given equation $11x+16y=100$, it follows that y must be less than 6: And therefore no whole numbers can be found to answer the question.

6. Given $17x+19y=2000$; to find all the values of x and y in affirmative whole numbers.

By transposition we get $17x=2000-19y$

And dividing by 17 gives $x=117+\frac{11-2y}{17}-y$:

Now it is evident that $\frac{11-2y}{17}$, the fractional part of the quotient, cannot be made a positive integer if y is a positive integer, whatever be its value,

therefore put $\frac{11-2y}{17}=-a$

Then, by reduction, $2y=17a+11$, and $y=8a+\frac{a+1}{2}+5$; where, if $a=1$, then $y=14$ the least affirmative value; and the corresponding or greatest value of x is 102.

The next value for a which gives $\frac{a+1}{2}$ an integer is 3, this being substituted, and we get $y=31$, whence $x=83$: now the difference between

102 and 83, the two values of x, is 19 the coefficient of y; and the difference of the two values of y is 17, the coefficient of x; therefore by constantly adding 17 to the last value of y, and subtracting 19 from that of x, we get all the other integral values.

$x = 102$	83	64	45	26	7
$y = 14$	31	48	65	82	99.

7. Suppose it is required to find integral values of x and y in the equation $9x + 15y = 100$.

Then since the coefficients 9 and 15 are divisible by 3, the sum $9x + 15y$ and also its equal 100 must be divisible by 3, whatever be the values of x and y; but 100 is not divisible by 3 without a remainder; therefore in this, and all similar cases, the unknown quantities x and y cannot be found in whole numbers.

8. How many different ways it is possible to pay 100£. with 7 shilling pieces, and dollars at 4*s.* 3*d.* each?

$$\left.\begin{array}{rcr} 100£. & = & 24000 \\ 7s. & = & 84 \\ 4s.\ 3d. & = & 51 \end{array}\right\} \text{pence.}$$

Let x denote the number of dollars, and y that of the 7*s.* pieces:

Then.............................. $51x + 84y = 24000$

Or, dividing by 3............ $17x + 28y = 8000$

whence......... $x = \dfrac{8000 - 28y}{17} = 470 + \dfrac{10 - 11y}{17} - y.$

Put $\dfrac{10 - 11y}{17} = -a$; then $10 - 11y = -17a$; whence $y = a + \dfrac{6a + 10}{11}$;

Now, let $\dfrac{6a + 10}{11} = b$; then $a = b + \dfrac{5b - 4}{6}$.

And making $\dfrac{5b - 4}{6} = c$, we have $b = c + \dfrac{c + 4}{5}$: Now if $c = 1$, then $b = 2$, and $a = 2$, whence $y = 4$ the least affirmative value of y; and the corresponding or greatest value of x is 464.

Hence, by adding 51 to the value of y, and subtracting 84 from that of x, we get the following answers, being 6 in number:

$x = 464$	380	296	212	128	44
$y = 4$	55	106	157	208	259.

9. To find a whole number which being divided by 15 shall leave 7, but when divided by 19 the remainder shall be 9.

If x be the required number; then $\frac{x-7}{15}$, and $\frac{x-9}{19}$ must be whole numbers, by the nature of the question.

Let $\frac{x-7}{15}=a$ (an integer); then $x-7=15a$, and $x=15a+7$, which being put for x in the second expression $\frac{x-9}{19}$, and we have $\frac{15a-2}{19}$ a whole number:

Now put $\frac{15a-2}{19}=b$; then $15a-2=19b$, and $a=b+\frac{4b+2}{15}$:

Again, make $\frac{4b+2}{15}=c$; then $4b+2=15c$, and $b=3c+\frac{3c-2}{4}$:

Next, let $\frac{3c-2}{4}=d$; and we get $c=d+\frac{d+2}{3}$:

Lastly, make $\frac{d+2}{3}=h$; then $d=3h-2$.

Therefore.......... $d=3h-2$

$c=d+h=4h-2$

$b=3c+d=12h-6+3h-2=15h-8$

$a=b+c=15h-8+4h-2=19h-10$

$x=15\,(19h-10)+7=285h-143$: Where h may be any affirmative integer whatever; consequently if it be 1, the value of x will be the least possible (viz. 142).

10. To find in what year of Christ the cycle of Indiction was 10, the Golden number or Lunar cycle 10, and the cycle of the Sun 8.

These Periods are found thus; Add 3, 1, and 9 to the year, and divide the sums by 15, 19, and 28, respectively; then the remainders will be the cycles.

Let x denote the year:

$$\text{Then}\quad \left.\begin{array}{lll} \frac{x+3-10}{15} & \text{or} & \frac{x-7}{15} \\[2ex] \frac{x+1-10}{19} & \text{or} & \frac{x-9}{19} \\[2ex] \frac{x+9-8}{28} & \text{or} & \frac{x+1}{28} \end{array}\right\}\ \text{must each be a whole number.}$$

By the preceding example, if $x = 285h - 143$, then $\frac{x-7}{15}$, and $\frac{x-9}{19}$ will be whole numbers; therefore substituting that value in the other expression gives $\frac{285h - 143 + 1}{28}$ or $\frac{285h - 142}{28}$, a whole number; put this $= a$,

Then $285h - 142 = 28a$, and $a = \frac{285h - 142}{28} = 10h + \frac{5h-2}{28} - 5$.

Let $\frac{5h-2}{28} = b$ (an integer); then $h = 5b + \frac{3b+2}{5}$:

Again, putting $\frac{3b+2}{5} = c$, gives $b = c + \frac{2c-2}{3}$:

And if $\frac{2c-2}{3} = d$, then $c = d + \frac{d+2}{2}$, where if $d = 0$, then $c = 1$, and $b = 1$, which gives $h = 6$, and $x = 1567$ the required year: which is also the least possible integral value of x.

11. Suppose the qualities of three ingredients are denoted by 10, 15, and 16; how many pounds of each must be taken to make a mixture of 80*lb.* with the quality 12?

Or, if 10, 15, and 16 pence are the prices per pound; what quantity of each will make a mixture of 80*lb.* at 12 pence per *lb.*?

Let x, y, and z denote the respective numbers of pounds:

Then.......... $x + y + z = 80$

And $10x + 15y + 16z = 80 \times 12 = 960$

From the last equation subtract 10 times the first,

$$\begin{array}{rl} & 10x + 15y + 16z = 960 \\ & \underline{10x + 10y + 10z = 800} \\ \text{there remains.......} & \phantom{10x + {}} 5y + 6z = 160 \end{array}$$

whence $y = \frac{160 - 6z}{5} = 32 - z - \frac{z}{5}$:

Let $\frac{z}{5} = a$; then $z = 5a$

Therefore $y = 32 - 6a$

And $x = 48 + a$ (or $80 - (32 - 6a) - 5a$):

Here a may be any positive number whatever less than $\frac{32}{6}$, and therefore the problem is unlimited. But if the values of the unknown quantities are restricted to integers, then, expounding a by 1, 2, 3, 4, and 5, we get the 5 following answers, which are all the question admits of in whole numbers.

$x=5$	10	15	20	25
$y=26$	20	14	8	2
$z=49$	50	51	52	53.

12. **How much gold of 15, of 17, and of 22 carats fine, must be mixed with 5 oz. of 18 carats fine, so that the composition may be 20 carats fine?**

If x, y, and z, are the respective quantities, and $a = 5$ oz. Then from the same principles upon which the preceding solution is founded, we shall have $15x + 17y + 22z + 18a = 20\ (x + y + z + a)$

or $15x + 17y + 22z + 18a = 20x + 20y + 20z + 20a$;

whence $2z = 5x + 3y + 2a$.

Now it is manifest without farther process, that the number of answers will be indefinite, for x and y may have any positive values; but if they are whole numbers, both must be even, or both odd, to give z an integer also: Thus, let $x = 8$ oz. and $y = 2$ oz. then $z = \frac{40 + 6 + 10}{2} = 28$ oz. And if $x = 5$, and $y = 5$; then $z = \frac{25 + 15 + 10}{2} = 25$ oz. &c.

13. **Suppose a mass of Gold, another of Copper, and a third which is a mixture of those metals, when separately immersed in the same vessel filled with water, expell 8.2, 17.9 and 11.5 ounces of the fluid, respectively: now if each mass weighs 160 *oz*. what quantity of copper is in the compound mass?**

The specific gravities of the metals will be reciprocally as the numbers 8.2, 17.9, and 11.5, or as 82, 179, and 115, which, therefore will denote the *rates* of the two simples, and the compound.

Let x be the copper, and y the gold in the compound mass,

Then $x + y = 160$

And $179x + 82y = 115x + 115y$

or............ $64x = 33y$.

But $y = 160 - x$

Therefore $64x = 33\ (160 - x) = 5280 - 33x$

And $97x = 5280$, or $x = 54.4$ oz. the quantity required.

In this manner it is said Archimedes discovered the quantity of alloy in a Crown that Hiero King of Sicily had ordered to be made with gold.

The Learner will perceive that the three last Examples belong to the Rule of *Alligation* in Arithmetic.

14. Suppose 960 troops are to be drawn up in three columns of march with 11, 14, and 19 men in front, respectively; now, how many different ways can this be done without any broken rank; and what number of ranks will each column consist of when their depths are the nearest possible equal to each other?

If x, y, and z denote the number of ranks in the respective columns,

then $11x+14y+19z=960$

And dividing the whole equation by 11 gives $x+y+\frac{3y}{11}+z+\frac{8z}{11}=87+\frac{3}{11}$

whence $x+y+z=87+\frac{3-3y-8z}{11}$.

Now, if y and z are any positive whole numbers, the expression $\frac{3-3y-8z}{11}$ must be negative;

therefore let......... $\frac{3-3y-8z}{11}=-a$

Then................. $3-3y-8z=-11a$

and............. $3y=11a+3-8z$

or...... $y=3a+\frac{2a}{3}+1-2z-\frac{2z}{3}$

that is $y=3a+1-2z+\frac{2a-2z}{3}$.

Let....... $\frac{2a-2z}{3}=b$

then $2a-2z=3b$

and $z=a-b-\frac{b}{2}$:

But it appears from the equation $y=3a+\frac{2a}{3}+1-2z-\frac{2z}{3}$, that a must be 3, or some multiple of 3, to make $\frac{2a}{3}$ a whole number; therefore to obtain the least integral value of z, make $a=3$, and $b=0$,

Then $z=a$

whence $y=a+1$; and substituting these values in the original equation, we get $x=86-3a$

Consequently, if a or $z=3$

then $y=4$

And $x=77$:

And the other values of y and x are found by constantly adding 11 (the coeffic [illegible]) [illegible] value of y, and subtracting 14 (the coefficient of y) from that of x. In this manner, by making z equal to 6, 9, 12, 15, &c. the multiples of 3, we get the following 28 answers:

$z=$	3						6					9				
$y=$	4	15	26	37	48	59	7	18	29	40	51	10	21	32	43	54
$x=$	77	63	49	35	21	7	68	44	30		2	59	45	31	17	3

$z=$	12				15			18			21	
$y=$	13	24	35	46	16	27	38	19	30	41	22	33
$x=$	50	36	22	8	41	27	13	32	18	4	23	9

Hence it appears that 23, 22, and 21 are the ranks in the respective columns when their depths are nearest alike.

And a similar method of solution may be followed when more than three unknown quantities are concerned. But different expedients will present themselves in the course of practice.

OF DIOPHANTINE PROBLEMS.

134. THESE are another kind of indeterminate Problems, called Diophantine, from *Diophantus* of Alexandria, an ancient Greek Mathematician who left a work on the subject, which chiefly relates to square and cube numbers. The Problems, for the most part, are of an abstruse nature, and do not seem to admit of any general method of solution. We shall subjoin a few easy examples in order to give the learner an idea of this part of Analysis.

Examples.

1. To divide a given number n into two such parts that the difference of their squares may be a given square a^2.

If x be the least part, then $n-x$ is the other:

And $(n-x)^2-x^2=n^2-2nx$ is the difference of their squares:

Whence, by the question, $n^2-2nx=a^2$

or $2nx=n^2-a^2$; and $x=\frac{n^2-a^2}{2n}$ the *least* part:

and $n - \frac{n^2-a^2}{2n} = \frac{n^2+a^2}{2n}$ the *greater*.

For $\left(\frac{n^2+a^2}{2n}\right)^2 - \left(\frac{n^2-a^2}{2n}\right)^2 = a^2$.

Here a^2 must be less than n^2, otherwise the problem is impossible.

Suppose $n=3$, and $a=2$; then $\frac{n^2-a^2}{2n} = \frac{5}{6}$; and $\frac{n^2+a^2}{2n} = \frac{13}{6}$: and the two parts are $\frac{5}{6}$ and $2\frac{1}{6}$.

2. To divide a given number n into two such parts that the sum of their squares shall be a square.

If x denotes one part, then $n-x$ will be the other:

And $(n-x)^2+x^2$ or $n^2-2nx+2x^2$ must be a square number: let its root be $n-ax$;

Then $n^2-2nx+2x^2 = (n-ax)^2 = n^2-2anx+a^2x^2$;

whence, by reduction, $x = \frac{2n-2an}{2-a^2}$ one part:

And $n - \frac{2n-2an}{2-a^2} = \frac{2an-a^2n}{2-a^2}$ the other, where a may be any assumed fraction less than unity.

Suppose the given number $(n)=8$; and let $a=\frac{1}{2}$;

Then $\frac{16-8}{2-\frac{1}{4}} = 4\frac{4}{7}$ one part; and $\frac{8-2}{2-\frac{1}{4}} = 3\frac{3}{7}$ the other.

3. To find two square numbers whose difference shall be a given number d.

Let $\frac{1}{2}x+\frac{1}{2}y$, and $\frac{1}{2}x-\frac{1}{2}y$, denote the roots of the required squares;

$$\begin{array}{rl}
\text{Then } (\frac{1}{2}x+\frac{1}{2}y)^2 = & \frac{1}{4}x^2+\frac{1}{2}xy+\frac{1}{4}y^2 \\
(\frac{1}{2}x-\frac{1}{2}y)^2 = & \frac{1}{4}x^2-\frac{1}{2}xy+\frac{1}{4}y^2 \\
\hline
\text{difference} & \qquad xy
\end{array}$$

Therefore $xy=d$ (by the question). Hence it appears that x and y may be any two unequal numbers whose product is the difference d: for should they be equal, then $\frac{1}{2}x-\frac{1}{2}y=0$.

Let $d=5$, $x=5$, and $y=1$; then $5\times1=5$:

$$\left.\begin{array}{l}
\text{And } \left(\frac{5}{2}+\frac{1}{2}\right)^2 = 9 \\
\left(\frac{5}{2}-\frac{1}{2}\right)^2 = 4
\end{array}\right\} \text{are two squares whose difference is 5.}$$

4. To divide a given square number s^2 into two other square numbers.

If x^2 be one of the squares, the others must be $s^2 - x^2$.

Assume $(nx - s)^2 = s^2 - x^2$: then $n^2x^2 - 2nsx + s^2 = s^2 - x^2$:

And, by reduction, we get $x = \left(\frac{2ns}{n^2+1}\right)$; therefore $x^2 = \left(\frac{2ns}{n^2+1}\right)^2$ one of the required squares; and $s^2 - \left(\frac{2ns}{n^2+1}\right)^2 = \left(\frac{sn^2 - s}{n^2+1}\right)^2$ the other; where n may be any number whatever, unity excepted.

Suppose 4 is the given square $(= s^2)$; and let $n = 3$:

Then $\left(\frac{2ns}{n^2+1}\right)^2 = \frac{36}{25}$ one square; and $\left(\frac{sn^2 - s}{n^2+1}\right)^2 = \frac{64}{25}$ the other, their sum being $\frac{36}{25} + \frac{64}{25} = 4$.

5. To divide a given number consisting of two square numbers, into two other square numbers.

Suppose the given number is $a^2 + b^2$; and let $x + a$, and $nx - b$ denote the roots of the required squares.

Then $(x+a)^2 + (nx-b)^2 = a^2 + b^2$

or $x^2 + 2ax + a^2 + n^2x^2 - 2nbx + b^2 = a^2 + b^2$

whence, by reduction, $x = \frac{2nb - 2a}{n^2+1}$:

Therefore $x + a = \frac{2nb + n^2a - a}{n^2+1}$; and $nx - b = \frac{bn^2 - 2na - b}{n^2+1}$:

And the two required squares, $\left(\frac{2nb + n^2a - a}{n^2+1}\right)^2$, and $\left(\frac{bn^2 - 2na - b}{n^2+1}\right)^2$, where n may be any assumed number except 1.

If the given number be 13 $(4+9)$; then $a = 2$, and $b = 3$: and let $n = 2$:

Then the two squares are $\left(\frac{12+8-2}{4+1}\right)^2 = \frac{324}{25}$, and $\left(\frac{12-8-3}{4+1}\right)^2 = \frac{1}{25}$: for $\frac{324}{25} + \frac{1}{25} = 13$.

Remark. Every number cannot be divided into two rational squares: For let the number 3 be proposed: then since it is not the sum of two integral squares, assume it equal to two fractional ones, or suppose $3 = \frac{a^2}{x^2} + \frac{b^2}{x^2} = \frac{a^2 + b^2}{x^2}$; now multiplying each side by 3 gives $9 = \frac{3a^2 + 3b^2}{x^2}$, therefore $3a^2 + 3b^2$ must be a square, but no two integral numbers can be found, such, that 3 times the sum of their squares is a square number; whence it appears that 3 is not resolvable into two squares. And Euler's conclusion in his Algebra is, that when a whole number is not the sum of two integral squares, it cannot be the sum of two fractional ones.

6. To find two numbers whose sum shall be equal to the sum of their cubes.

This admits of one solution in integers, viz. when each of the numbers is 1; for $1 + 1 = 1^3 + 1^3$. But other answers may be found in fractions thus,

Let x and y denote the two numbers; then $x + y = x^3 + y^3$; and dividing the whole equation by $x + y$, gives $1 = \frac{x^3 + y^3}{x + y} = x^2 - xy + y^2$.

Put $nx = y$; then $1 = x^2 - nx^2 + n^2x^2$, whence $x^2 = \frac{1}{n^2 - n + 1}$ which must be a square number because x^2 is a square; and consequently $n^2 - n + 1$ must also be a square.

Let $n^2 - n + 1 = a^2$; then $n^2 - n = a^2 - 1$;
and completing the square, $n^2 - n + \frac{1}{4} = a^2 - 1 + \frac{1}{4} = a^2 - \frac{3}{4}$,
whence $n - \frac{1}{2} = \sqrt{(a^2 - \frac{3}{4})}$:

The problem is now reduced to that of making $a^2 - \frac{3}{4}$ a rational square number, which is done by *Examp.* 3: for if we assume two numbers whose product is $= \frac{3}{4}$, half their sum will be the value of a:

Thus, let 1 and $\frac{3}{4}$ be assumed; then $1 \times \frac{3}{4} = \frac{3}{4}$ and half of $1 + \frac{3}{4}$ is $\frac{7}{8} = a$; therefore $a^2 = \frac{49}{64}$, and $\frac{49}{64} - \frac{3}{4} = \frac{1}{64}$:

Therefore $n - \frac{1}{2} = \sqrt{\frac{1}{64}} = \frac{1}{8}$, whence $n = \frac{5}{8}$:

Now, $x^2 = \frac{1}{n^2 - n + 1} = \frac{64}{49}$, and $x = \frac{8}{7}$; therefore y or $nx = \frac{5}{8} \times \frac{8}{7} = \frac{5}{7}$. Hence $\frac{8}{7}$ and $\frac{5}{7}$ are two fractions answering the conditions of the problem.

If 3 and $\frac{1}{4}$ are assumed; then $3 \times \frac{1}{4} = \frac{3}{4}$; and we get $\frac{15}{13}$ and $\frac{8}{13}$, which are two other fractions whose sum is equal to the sum of their cubes.

7. Should it be required to find two numbers whose *difference* is equal to the *difference* of their cubes:

Then $x - y = x^3 - y^3$; and $1 = \frac{x^3 - y^3}{x - y} = x^2 + xy + y^2$; and putting $nx = y$, as before, we get $x^2 = \frac{1}{n^2 + n + 1}$: now assuming $n^2 + n + 1 = a^2$ (a square), gives $n = \sqrt{\left(a^2 - \frac{3}{4}\right)} = \frac{1}{2}$, where the root $\sqrt{\left(a^2 - \frac{3}{4}\right)}$ must be *less* than $1\frac{1}{2}$, but *greater* than $\frac{1}{2}$.

Let 2 and $\frac{3}{8}$ be assumed, their product being $2 \times \frac{3}{8} = \frac{6}{8}$ or $\frac{3}{4}$: then $\frac{2 + \frac{3}{8}}{2} = \frac{19}{16} = a$, and $a^2 = \frac{361}{256}$; therefore $n = \sqrt{\left(\frac{361}{256} - \frac{3}{4}\right)} - \frac{1}{2} = \frac{5}{16}$: now $\frac{1}{n^2 + n + 1} = \frac{256}{361} = x^2$, whence $x = \frac{16}{19}$; and nx or $y = \frac{5}{16} \times \frac{16}{19} = \frac{5}{19}$. Therefore $\frac{16}{19}$ and $\frac{5}{19}$ are two fractions whose difference is equal to the difference of their cubes. And in like manner other answers may be found.

The three following Theorems will frequently be found useful in problems which relate to square numbers.

1. If $2ab$, $a^2 - b^2$, and $a^2 + b^2$, denote the roots of three square numbers; then $(2ab)^2 + (a^2 - b^2)^2 = (a^2 + b^2)^2$. By this Theorem two square numbers may be found whose sum or difference shall be square numbers.

2. If d be any number; then $(d^2 + (d+1)^2) \times ((d+1)^2 + (d+2)^2) \pm (d+1)^2 \times 4$ will be two squares, whose roots are $2d^2 + 4d + 3$, and $2d^2 + 4d + 1$.

3. If a and b be any two numbers; then $(a^2 + b^2)^2 \pm (a^2 - b^2) \times 4ab$ will be two squares, the roots being $a^2 \pm 2ab - b^2$.

OF ARITHMETICAL PROGRESSIONS.

135. THE nature of Arithmetical proportion and progression has already been explained, (Arith. *Art.* 121). It is by the help of analysis however, that we must discover the different relations which the several terms have to one another: besides, algebraic formulæ are better adapted to practice, and more concise than verbal enunciations.

136. Let f be the first term of an arithmetical progression.
d the common difference of the terms.
l the last term.
n the number of terms.
s the sum of all the terms.

Then f, $f+d$, $f+2d$, $f+3d$, $f+4d$, &c. will be an ascending series or progression, (122 Arith.)

And f, $f-d$, $f-2d$, $f-3d$, $f-4d$, &c. a descending one.

Hence it appears that the last term is always $=f+(n-1)d$ in an ascending progression, and $f-(n-1)d$ in a descending one.

137. The sum of all the terms in an arithmetical progression is equal to the sum of the first and last terms multiplied by half the number of terms; *viz.* $s=(f+l)\frac{n}{2}$. (132 Arith.)

Let $f+f+d+f+2d+f+3d+f+4d=s$;

then $5f+10d=s$, *viz.* $(f+f+4d)\times\frac{5}{2}=s$, or $(f+l)\frac{n}{2}=s$.

And if the number of terms be 6, the last term will be $f+5d$;

And the sum $=6f+15d=s$, or $(f+f+5d)\frac{6}{2}=s$, that is $(f+f+(n-1)d)\frac{n}{2}$, or $(f+l)\frac{n}{2}=s$.

Now, from the equations $f+(n-1)d=l$, $(f+f+(n-1)d)\frac{n}{2}=s$, and $(f+l)\frac{n}{2}=s$, the following theorems or formulæ are readily obtained, where it is to be noted, that when the progression is descending, the signs of the terms affected with d must be changed, or f taken for l, and *vice versâ*; these forms being adapted to an ascending series.

138. $f=l-nd+d=\sqrt{(-2sd+l^2+dl+\frac{1}{4}d^2)}+\frac{1}{2}d=\frac{s}{n}+\frac{1}{2}d-\frac{1}{2}nd=\frac{2s}{n}-l.$

$$d=\frac{l-f}{n-1}=\frac{l^2-f^2}{2s-l-f}=\frac{2ln-2s}{n^2-n}=\frac{2s-2nf}{n^2-n}.$$

$$l=f+nd-d=\frac{s}{n}+\frac{1}{2}nd-\frac{1}{2}d=\sqrt{(2sd+f^2-df+\frac{1}{4}d^2)}-\frac{1}{2}d=\frac{2s-nf}{n}.$$

$$s=\frac{nf+nl}{2}=(f+\tfrac{1}{2}nd-\tfrac{1}{2}d)n=(l-\tfrac{1}{2}nd+\tfrac{1}{2}d)n=\frac{l^2-f^2+dl+df}{2d}.$$

$$n=\frac{2s}{f+l}=\frac{l-f}{d}-1.$$

Let $\frac{2f\backsim d}{2d}=r$, then

$n=\sqrt{\left(\frac{2s}{d}+r^2\right)}-r$ when $2f$ is *greater* than d.

$n=\sqrt{\frac{2s}{d}}$ ……………… when $2f=d$.

$n=\sqrt{\left(\frac{2s}{d}+r^2\right)}+r$ when $2f$ is *less* than d.

139. A few examples will show the use of these expressions.

1. Required the sum of the series $1+3+5+7+$ &c. continued to 20 terms?

Here $f=1$, $d=2$, and $n=20$, which substituted in the form $s=(f+\frac{1}{2}nd-\frac{1}{2}d)n$ gives $s=(1+20-1)\ 20$, or $20\times20=400$ the sum required.

Hence it appears, that the sum of the odd numbers $1+3+5+7$ &c. continued to n terms, is always $=n^2$.

2. What is the 17*th*. term of the series 10, $9\frac{2}{3}$, $9\frac{1}{3}$, 9, &c.

In this progression $f=10$, $d=\frac{1}{3}$, $n=17$; and the corresponding expression is $l=f+nd-d$ which, when the signs of the terms $+nd-d$ are changed (the series being a descending one) becomes $l=f-nd+d$, or $l=10-17\times\frac{1}{3}+\frac{1}{3}=4\frac{2}{3}$ the required term.

3. Two detachments, distant from each other 39 leagues, and both designing to occupy an advantageous post equidistant from each other's camp, set out at different times; the first detachment increasing every day's march one league and a half, and the second detachment decreasing each day's march 2 leagues: both detachments arrive at the same time; the first after 5 days march, and the second after 4 days march: What is the number of leagues marched by each detachment each day?

The whole distance marched by each detachment is $\frac{39}{2}=19\frac{1}{2}$ leagues. Therefore, for the first detachment, we have $d=1\frac{1}{2}$, $n=5$, and $s=19\frac{1}{2}$; and to find the first term or distance marched the first day, the expression is $f=\frac{s}{n}+\frac{1}{2}d-\frac{1}{2}nd$, or $\frac{19\frac{1}{2}}{5}+\frac{3}{4}-3\frac{3}{4}=\frac{9}{10}$ of a league; whence $\frac{9}{10}$, $2\frac{4}{10}$, $3\frac{9}{10}$, $5\frac{4}{10}$, $6\frac{9}{10}$ are the respective distances marched each day.

And the same theorem or expression answers for the distance marched by the other detachment on the first day when the signs of the two last terms are changed; for $d=2$, $n=4$, and $s=19\frac{1}{2}$, whence $f=\frac{s}{n}-\frac{1}{2}d+\frac{1}{2}nd=\frac{19\frac{1}{2}}{4}-1+4=7\frac{7}{8}$ leagues the first day's march; therefore the distances are $7\frac{7}{8}$, $5\frac{7}{8}$, $3\frac{7}{8}$, $1\frac{7}{8}$.

4. A detachment of dragoons being sent after a deserter, marched the first day 9 miles, the second 19, the third 29, and so on, increasing the distance 10 miles each day; in what time did they overtake him, supposing he travelled at the rate of 34 miles a day?

This is readily answered by means of the expression $s=\frac{nf+nl}{2}$; for the number of days will be the number of terms, and $34n$ is equal to s the whole distance travelled; therefore substituting 9 the first term, for f, we have $\frac{9n+nl}{2}=34n$, or $\frac{9+l}{2}=34$, whence $l=59$ the number of miles which the detachment marched on the last day; consequently they overtook the deserter in 6 days.

5. A company of foot leave London for Plymouth, and at the same time a party of horse are ordered from Plymouth to London; the foot march 14 miles the first day, 13 the second, 12 the third, &c. constantly lessening each day's march 1 mile; but the horse travel 8 miles the first day, and increase their march 4 miles every day; what distance will each party have travelled when they meet, if Plymouth is 217 miles from London?

In this example we have the sum of a descending and an ascending progression	$=217$
The first term of one progression.................	$=14$
The common difference of the terms..................	$=1$
The first term of the other series......................	$=8$
And common difference	$=4$

And the number of terms (or days) in each progression is the same.

The formula adapted to this case is $n=\sqrt{\left(\frac{2s}{d}+r^2\right)}-r$: therefore if 8 and 4 are substituted for f and d we get $\frac{2f-d}{2d}=\frac{16-4}{8}=1\frac{1}{2}=r$: and putting x for the miles travelled by the foot, the distance travelled by the horse will be $217-x=s$; whence $n=\sqrt{\left(\frac{2s}{d}+r^2\right)}-r=\sqrt{\left(\frac{434-2x}{4}+\frac{9}{4}\right)}-1\frac{1}{2}$ the number of terms, or days travelled by the party of horse.

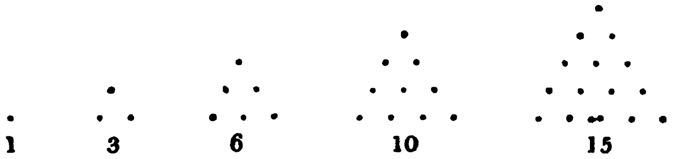
But the expression $n=\sqrt{\left(\frac{2s}{d}+r^2\right)}-r$ is derived from $f=\frac{s}{n}-\frac{1}{2}nd+\frac{1}{2}d$, which becomes $f=\frac{s}{n}+\frac{1}{2}nd-\frac{1}{2}$ when f is the first term of a descending progression, and in this case $r=\frac{2f+d}{2d}$, and $n=r-\sqrt{\left(r^2-\frac{2s}{d}\right)}$.

Now, in the descending series, $f=14$, and $d=1$; therefore $\frac{2f+d}{2d}=\frac{28+1}{2}=14\frac{1}{2}=r$, and $n=r-\sqrt{\left(r^2-\frac{2s}{d}\right)}=14\frac{1}{2}-\sqrt{(210\frac{1}{4}-2x)}$ the number of terms, or days travelled by the foot:

Consequently we have this equation $\sqrt{\left(\frac{434-2x}{4}+\frac{9}{4}\right)}-1\frac{1}{2}=14\frac{1}{2}-\sqrt{(210\frac{1}{4}-2x)}$, which reduced gives $9x^2+3926x=355663$, whence $x=77$ the miles travelled by the company of foot; therefore $217-77=140$ the distance for the horse.

OF FIGURATE NUMBERS.

140. THESE numbers result from the sums of arithmetical series, and are called *figurate* because the units in each term may be so disposed as to represent a geometrical figure or diagram, as a triangle, a square, or a pentagon, &c. Thus 1, 3, 6, 10, &c., are a series of triangular numbers:



Let any number of units be disposed as in *fig.* 1, where the perpendicular ranks are one more in number than the horizontal ones. And draw the diagonal line AB to divide the number of terms into two equal parts.

fig. 1.

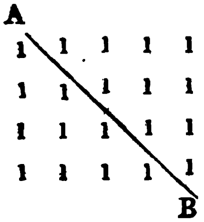


fig. 2.

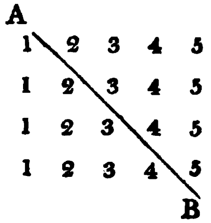


fig. 3.

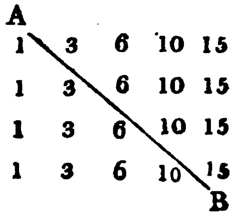


fig. 4.

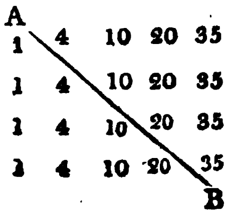
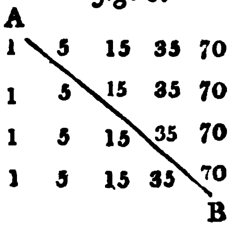


fig. 5.



Then 1, 1, 1, 1, &c. (*fig.* 1.) are called a series of the first order:

And the sums of the units on the left of the line AB form the progression 1, 2, 3, 4, &c. or a series of the second order. These being disposed as in *fig.* 2. the sums of the horizontal rows on the left of AB constitute the series 1, 3, 6, 10, &c. or the third order:

Thus 1 .... = 1
1 + 2 .... = 3
1 + 2 + 3 .... = 6
1 + 2 + 3 + 4 = 10, &c.

Again, the sums of the horizontal ranks on the left of the line AB in *fig.* 3. form the fourth order:

for 1 .... = 1
1 + 3 .... = 4
1 + 3 + 6 .... = 10
1 + 3 + 6 + 10 = 20, &c.

And so on, for the several orders.

Order
141. 1 .... 1, 1, 1, 1, 1, &c.
2 .... 1, 2, 3, 4, 5, &c.
3 .... 1, 3, 6, 10, 15, &c.
4 .... 1, 4, 10, 20, 35, &c.
5 .... 1, 5, 15, 35, 70, &c.

Hence it appears that the last term in any order is always equal to the sum of all the terms in the next inferior one:

Thus the *5th.* term in the *3d.* order is 15, which is equal to 1 + 2 + 3 + 4 + 5 in the *2d.* order.

142. The line AB in *fig.* 1, divides the sum of all the terms into two equal parts; but in *fig.* 2, the sum of all the terms

$(1 + 3 + 6$, &c.) on the left of AB is $= \frac{1}{3}$ of the sum of all the terms in the figure; for each vertical row on the right hand of AB is double the corresponding horizontal row on the left: thus $3 + 3 =$ twice $1 + 2$; $4 + 4 + 4 =$ twice $1 + 2 + 3$, &c. In like manner, the sum of all the terms on the left of AB, *fig.* 3, is $= \frac{1}{4}$ of all the terms in that figure. And in *fig.* 4, the sum on the left is $\frac{1}{5}$ of the whole, &c.

143. Now, let n denote the number of terms in a vertical row, or the number of horizontal ranks; then $n + 1$ will be the number of terms in an horizontal rank, and in *fig.* 2, $n + 1$ is the last term in that rank or series: therefore (137) $\frac{n+1}{2} \times (n + 2)$ will be the sum of the series $1 + 2 + 3$, &c, or of all the terms in that rank, which multiplied by n the number of horizontal ranks, is $\frac{n+1}{2} \times (n + 2) \times n$, the sum of all the terms in the figure, and $\frac{1}{3}$ of that sum or $\frac{n+1}{2} \times (n + 2) \times n \times \frac{1}{3} = n \times \frac{n+1}{2} \times \frac{n+2}{3}$ is the sum of all the terms on the left of AB, or the sum of the series $1 + 3 + 6 + 10$, &c. continued to n terms.

144. To find the sum of the series $1 + 3 + 6 + 10$, &c. to $n + 1$ terms, substitute $n + 1$ for n, and $n \times \frac{n+1}{2} \times \frac{n+2}{3}$ becomes $\frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3}$ the sum of all the terms in an horizontal rank (*fig.* 3), which multiplied by n the number of ranks, gives $\frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times n$ the sum of all the terms which compose the figure, and $\frac{1}{4}$ of this is $\frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n}{4}$, or $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$ the sum of all the terms on the left of AB, or sum of the series $1 + 4 + 10 + 20$, &c. continued to n terms.

Hence it appears that $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}$. &c. continued to 5 factors will be the sum of the series which is the *5th.* order of figurate numbers: and 6 factors give the sum in the *6th.* order, &c.

145. These series are useful in computing the number of cannon shot in a pile. The piles are usually triangular, square, or oblong. The triangular pile has an equilateral triangle for its base, and ends in a ball at top; and the several layers or courses of shot form the series 1, 3, 6, 10, 15, &c. from the top downward, the last term being the number of shot in the course next the ground.

Now the series $1+3+6+10$, &c. is the third order, and its sum or the number of shot in a triangular pile is $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}$, where n is the number of courses or the number of shot in the side of the base.

Suppose the number of courses in a triangular pile or pyramid is 40: then $\frac{40}{1}\times\frac{40+1}{2}+\frac{40+2}{3}=11480$ the number of shot in such a pile.

146. The square pile is a pyramid having a square for its base, and a single ball at the top, this ball with the successive courses downward constitute the series of squares, 1, 4, 9, 16, 25, &c. the last term being the number of shot in the bottom course.

$$\begin{array}{ll} \text{The series of squares} & 1+4+9+16+25,\ \text{\&c. to } n \text{ terms}\\ \left.\begin{array}{l}\text{may be resolved into}\\ \text{the two series.........}\end{array}\right\} & \begin{array}{r} 1+3+6+10+15,\ \text{\&c. to } n \text{ terms}\\ \underline{1+3+6+10,\ \text{\&c. to } n-1 \text{ terms}}\end{array}\\ \text{sum} & \underline{1+4+9+16+25,\ \text{\&c.}} \end{array}$$

The sum of $1+3+6+10$, &c. to n terms is $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}$, and putting $n-1$ for n, gives $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n+1}{3}$ the sum to $n-1$ terms; and the sum of both these expressions, or $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n+1}{3}$ $=\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1}{3}$ is the sum of the series of squares $1+4+9+16$, &c. continued to n terms.

Suppose 30 shot in the side of the bottom course, then $n=30$ the number of courses, and $\frac{n}{1}\times\frac{n+1}{1}\times\frac{2n+1}{3}=9455$ the number of shot in the pile.

147. The oblong pile stands on a rectangular base; the number of shot in any course being found by multiplying the number of shot in one of its sides by the number of shot in the other side: and the whole pile is composed of a series of rectangular courses, the sides each diminishing by 1 from the base upwards; therefore if d be the difference of the shot in the sides of any course, the pile will end at top in a rank of $d+1$ balls.

Thus, if the sides of the bottom course contain 12 and 7 shot,

$$\text{Then}\quad \begin{array}{r}12\times 7=84\\ 11\times 6=66\\ 10\times 5=50\\ 9\times 4=36\\ 8\times 3=24\\ 7\times 2=14\\ 6\times 1=\ \ 6\end{array}\quad \begin{array}{l}\\ \\ \text{are the shot in}\\ \text{the successive courses.}\\ \\ \\ \\ \end{array}$$

The whole sum may be found by resolving the series $6+14+24+36$, &c. into two other series,

$$\begin{array}{ll}\text{thus} & 1+\ \ 4+\ \ 9+16+25\ \text{\&c.}\\ & 5+10+15+20+25\ \text{\&c.}\\ \hline \text{sum} & 6+14+24+36+50\ \text{\&c.}\end{array}$$

The sum of the squares $1+4+9$ &c. continued to n terms is $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1}{3}$. The last term of the series $5+10+15$ &c. continued to n terms is nd; and the sum of the same series to n terms is $n\times\frac{nd+d}{2}$. (137.)

Therefore the sum of both series, $\frac{n}{1}\times\frac{n+1}{2}\times\frac{2n+1}{3}+n\times\frac{nd+d}{2}$, or $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1+3d}{3}$ is the sum of the series of products continued to n terms, where n is the number of courses, or the number of shot in the least side of the bottom course, and d the difference of the number of shot in that side and the other.

Let the sides of the bottom course contain 32 and 25 shot;

Then $n=25$, and $d=7$, and $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1+3d}{3}=7800$ the number of shot in the pile.

148. If the pile is broken, find the number of shot deficient, and also the whole number it would contain, supposing it complete, then the difference will be the shot remaining.

Suppose the shot in the sides of the bottom course of a broken pile are 32 and 25, and in the upper course 23 and 16, then the shot in the sides of the next course would be 22 and 15; therefore $n=15$ the number of courses deficient, and $d=7$; and $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1+3d}{3}=2080$ the number of shot deficient: now the number in the complete pile (found above) would be 7800; therefore $7800-2080=5720$ is the number of shot in the broken pile. And in this manner we may proceed when the broken pile is triangular, or square.

149. If s be the number of shot in a complete triangular pile; and we would find the number of courses or the number of shot in the side of the base; then $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}=\frac{n^3+3n^2+2n}{6}$ $=s$, or $n^3+3n^2+2n=6s$; and if $n+1$ be added to each side of this equation, we get $n^3+3n^2+3n+1=6s+n+1$: now n^3+3n^2+3n+1 is a cube whose root is $n+1$; therefore $n+1=(6s+n+1)^{\frac{1}{3}}$; and since the value of n is restricted to an integer, $6s+n+1$ will be the integral cube next greater than $6s$.

Let the number of shot in a complete triangular pile be 11480; then $11480\times 6=68880$, and the cube next greater is 68921 whose root is $41=n+1$, therefore $n=40$.

150. In the complete square pile we have $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{2n+1}{3}=$ $\frac{2n^3+3n^2+n}{6}=s$, or $n^3+1\frac{1}{2}n^2+\frac{1}{2}n=3s$, and adding $1\frac{1}{2}n^2+2\frac{1}{2}n+1$ to each side of the equation, gives n^3+3n^2+

$3n + 1 = 3s + 1\frac{1}{2}n^2 + 2\frac{1}{2}n + 1$, whence $n + 1 = (3s + 1\frac{1}{2}n^2 + 2\frac{1}{2}n + 1)^{\frac{1}{3}}$; and $n + 1$ will be equal to the root of the integral cube next greater than $3s$.

Suppose $s = 9455$; then $3s = 28365$, and the cube number next greater is 29791 whose root is $31 = n + 1$; whence $n = 30$ the number of courses.

Of GEOMETRICAL PROGRESSIONS.

151. Let f be the first term of a geometrical series of quantities:

r the common ratio of the terms:
l the last term of the series:
n the number of terms:
s the sum of all the terms.

Then $f, rf, r^2f, r^3f, \ldots . . r^{n-1}f$, is a geometrical progression or series of quantities, (146 Arith.)

Hence it appears that the series will be ascending, or descending according as the ratio or multiplier r is greater, or less than 1:

And that the last term is always $= r^{n-1}f = l$.

152. To find the sum of the progression:

$$\begin{array}{lcl} \text{Let } f + rf + r^2f + r^3f + r^4f \;\&\text{c} & = & s, \text{ this multiplied by } r \\ \text{gives } \underline{rf + r^2f + r^3f + r^4f + r^5f \;\&\text{c}} & = & \underline{rs} \\ \underline{-f \;\; . \;\; . \;\; . \;\; . \;\; + r^5f} & = & \underline{rs - s} \text{ the remainder} \end{array}$$

when the upper series &c. is subtracted from the lower:

But r^5f is the last term r^4f multiplied by r, or $r^{n-1}f \times r = r^nf$:

Therefore $-f + r^5f = r^nf - f = rs - s$

whence $\frac{r^nf - f}{r - 1} = s$ the sum of the series. This process is exactly similar to the arithmetical operation, (Arith. *Art.* 152.)

Now, from the equations, $r^{n-1}f = l$;

$$\frac{r^n f - f}{r - 1} = s;$$

we obtain the following theorems or formulæ:

153. $$f = \frac{l}{r^{n-1}} = s \times \frac{r-1}{r^n - 1} = s + rl - rs.$$

$$l = f \times r^{n-1} = \frac{rs - s + f}{r} = s \times \frac{r^n - r^{n-1}}{r^n - 1}.$$

$$s = f \times \frac{r^n - 1}{r - 1} = \frac{rl - f}{r - 1} = \frac{lr^n - l}{r^n - r^{n-1}}.$$

$$r = \frac{s - f}{s - l} = \left(\frac{l}{f}\right)^{\frac{1}{n-1}}.$$

Put the *logarithm* of $\frac{l}{f} = \mathrm{Q}$;

the *log.* of $\frac{sr - s + f}{f} = \mathrm{P}$;

the *log.* of $r = \mathrm{R}$; then

$$n = \frac{Q}{R} + 1 = \frac{P}{R}.$$

The *logarithmic* expressions for the value of n are derived from the method of raising powers by means of logarithms, explained in the Arithmetic, *Art.* 161, 187, thus;

Since $f \times r^{n-1} = l$, therefore $r^{n-1} = \frac{l}{f}$: and because the logarithm of any power of a number is equal to the logarithm of that number multiplied by the index or exponent denoting the power (187 Arith.) therefore $(n - 1)\ log.\ r = log.\ \frac{l}{f}$, that is, $n R - R = Q$, whence $n - 1 = \frac{Q}{R}$, or $n = \frac{Q}{R} + 1$.

And the other expression for n is found in the same manner; for $s \times \frac{r-1}{r^n - 1} = f$, which gives $r^n = \frac{sr - s + f}{f}$, whence $n \times log.\ r = log.\ \frac{sr - s + f}{f}$, or $n R = P$, therefore $n = \frac{P}{R}$.

154. Some Examples explaining the use of the preceding Theorems.

1. What is the sum of the progression $2+6+18+54$, &c. continued to 10 terms?

Here $f=2$, $r=3$, and $n=10$:

And $s=f\times\frac{r^n-1}{r-1}=2\times\frac{3^{10}-1}{3-1}=59048$ the *sum required.*

2. Required the sum of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ &c. continued to 8 terms?

Here $f=\frac{1}{2}$, $r=\frac{1}{2}$, and $n=8$:

Then $s=f\times\frac{r^n-1}{r-1}=\frac{1}{2}\times\frac{(\frac{1}{2})^8-1}{\frac{1}{2}-1}=\frac{1}{2}\times\frac{-\frac{255}{256}}{-\frac{1}{2}}=\frac{255}{256}$ the *answer.*

3. What is the sum of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$, &c. continued *ad infinitum?*

If the terms are supposed to be infinite in number, the last term must be $=0$: therefore $f=1$, $r=\frac{1}{3}$, and $l=0$:

And $s=\frac{rl-f}{r-1}=\frac{0-1}{\frac{1}{3}-1}=\frac{-1}{-\frac{2}{3}}=1\frac{1}{2}$ the *sum of the series.*

4. What is the sum of the series $\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}$ &c. infinitely continued?

In this progression $f=\frac{6}{10}$, $r=\frac{1}{10}$, and the last term $l=0$:

And $s=\frac{rl-f}{r-1}=\frac{0-\frac{6}{10}}{\frac{1}{10}-1}=\frac{-\frac{6}{10}}{-\frac{9}{10}}=\frac{6}{9}=\frac{2}{3}$ the *answer.*

This series is the decimal .6666 &c. or $\frac{2}{3}$ reduced to a decimal.

5. Required the vulgar fraction corresponding to the recurring decimal .363636, &c.

This decimal may be resolved into the series $\frac{36}{100}+\frac{36}{10000}+\frac{36}{1000000}$ &c. where $f=\frac{36}{100}$, $r=\frac{1}{100}$, and $l=0$:

$$\text{Then } s=\frac{rl-f}{r-1}=\frac{-\frac{36}{100}}{-\frac{99}{100}}=\frac{36}{99}=\frac{4}{11} \text{ the } \textit{answer}.$$

Hence, to find the vulgar fraction answering to a circulating decimal of this kind, make the figures which are repeated, the numerator, and the same number of nines, the denominator, and that will form the fraction.

Thus in the preceding example, 36 are the two figures repeated, which placed over two nines make $\frac{36}{99}$.

And if .7142857142 &c. be the decimal proposed, then $\frac{714285}{999999}$ or $\frac{5}{7}$ is the equivalent vulgar fraction.

6. To find the vulgar fraction answering to the decimal .41666, &c.

$$.41666\text{ \&c.}=\left\{\begin{array}{l} +\frac{41}{100} \\ +\frac{6}{1000}+\frac{6}{10000}, +\text{\&c.}\end{array}\right.$$

The sum of the series $\frac{6}{1000}+\frac{6}{10000}$ &c. is $\frac{6}{900}$

Therefore $\frac{41}{100}+\frac{6}{900}=\frac{375}{900}=\frac{5}{12}$ is the vulgar fraction sought.

7. What is the sum of the progression $1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+$ &c. continued *ad infinitum?*

Here $f=1$, $r=-\frac{2}{3}$, and $l=0$:

$$\text{And } s=\frac{rl-f}{r-1}=\frac{-1}{-1\frac{2}{3}}=\frac{3}{5} \text{ the } \textit{answer}.$$

8. Required the sum of the descending series $1-x+x^2-x^3+$ &c. infinitely continued?

In this progression $f=1$, $r=-x$, and $l=0$:

Therefore $s = \frac{rl - f}{r - 1} = \frac{-1}{-x-1} = \frac{1}{1+x}$ the *answer.*

For by actual division, $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \&c.$

9. If the first term of a series be 2187, the last 128, and the ratio $\frac{2}{3}$; what is the number of terms?

Here $f = 2187$, $l = 128$, and $r = \frac{2}{3}$.

$\frac{l}{f} = \frac{128}{2187}$ .... *log.* 2.107210
.... *log.* 3.339849

$\frac{128}{2187}$ *log.* — 2.767361 = Q.

$\frac{2}{3}$.... *log.* — 1.823909 = R.

$\frac{Q}{R} = \frac{-2.767361}{-1.823909} = 7$; and $n = \frac{Q}{R} + 1 = 7 + 1 = 8$ the number of terms.

The learner must remember that the *indices only* of the logarithms are negative; whence, in dividing the *log.* of Q by that of R, the *positive* 5 which is carried to the *negative* 7 make 2 *negative*, and therefore one logarithm is contained in the other 7 times.

But in this case the use of negative indices may be avoided by making the last term the first, and *vice versâ*, and taking the reciprocal of the ratio r: thus;

Let $f = 128$, $l = 2187$, and $r = \frac{3}{2}$:

$\frac{l}{f} = \frac{2187}{128}$ .... *log.* 3.339849
.... *log.* 2.107210

1.232639 = Q

$r = \frac{3}{2}$ *log.* 0.176091 = R

and $\frac{Q}{R} = \frac{1.232639}{0.176091} = 7$, and $n = 7 + 1 = 8$ the number of terms as before.

When the last term is = 0, the number of terms (n) must evidently be infinite.

10 Suppose the first term of a series to be 4, the ratio $\frac{1}{2}$, and the sum of the series 8; what is the number of terms?

Here l (the last term) $= \frac{rs - s + f}{f} = \frac{8 \times \frac{1}{2} - 8 + 4}{4} = \frac{0}{4}$ an indefinite, or infinitely small quantity; therefore (n) the number of terms must also be infinite.

We may also remark that when the number of terms are infinite, the expression $r = \left(\frac{l}{f}\right)^{\frac{1}{n-1}}$ will not give the value of the ratio r.

155. When the numbers are too great for the logarithmic tables, the value of n or number of terms may be found from actual multiplication, or the powers of the ratio r, thus;

Suppose the first term of a progression to be 7, the ratio 3, and the sum of the series $= 36611236207$: then from the expression $l = \frac{rs - s + f}{r}$ we get l the last term $= 24407490807$, which divided by 7 the first term, gives $3486784401 = 3^{n-1}$, now 3486734401 is the *20th.* power of 3, and therefore $n = 21$ the number of terms.

Of PERMUTATIONS and COMBINATIONS.

156. When a given number of things or quantities stand in any order or position, and that order is varied by changing the situation or place of any one of the quantities or things, it is called a Permutation.

Thus, one thing or quantity a is said to admit of one position only: But two things a and b can be varied, for a may stand first and b second, and *vice versâ*, thus *ab*
ba

And the variations or changes are $1 + 2$.

157. If the number of things are three, as a, b, c, then each may stand first two times while the other two change places, therefore 3 things can be varied 2×3, or $1 \times 2 \times 3$ times;

Thus *abc* *bac* *cab*
acb *bca* *cba*

158. Four things *a*, *b*, *c*, *d*, are capable of 6×4 or $1 \times 2 \times 3 \times 4$ permutations; for each may stand the first, or the last, 6 times in a successive order, the other three being varied as above:

Thus *abcd*
acbd
bacd
bcad
cabd
cbad
abdc
adbc
bdac
badc
dabc
dbac
adcb
acdb
dcab
dacb
cadb
cdab
bcda
bdca
cdba
cbda
dcba
dbca

159. And 5 things will admit of 24×5 or $1 \times 2 \times 3 \times 4 \times 5$ changes; for each may occupy the 5*th.* place 24 times successively. Hence it appears that the permutations in *n* things are $1 \times 2 \times 3 \times 4$ &c. continued to *n* factors.

Examples.

1. How many changes can be rung on 8 bells?

$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320$ the *answer.*

2. If 6 columns of troops are in order of march; how many times can that order be varied?

$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$, *answer.*

3. How many variations or changes can take place in the letters of the word *permutation?*

$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 = 39916800$ the *answer.*

160. By the *combinations* or *elections* of quantities or things we understand the different collections that can be formed out of them, without any regard to their order, as in permutations.

Thus, suppose *a*, *b*, *c*, are the quantities, and that each collection or combination consists of two of them, then *ab*, *bc*, and *ac* are three different combinations, no two being alike.

161. *To investigate the number of combinations.* First, suppose the number of things in each combination to be two: then if the number of quantities or things are only two (a and b) it is evident there can be but 1 combination, ab.

Next, let the quantities be a, b, c; then since c can be combined with each of the two former letters a and b, the number of combinations will be increased by 2; therefore the number of combinations of 2 quantities in 3 will be $1+2$:

thus ab, ac, bc.

When the quantities or things are augmented to four, a, b, c, d, the number of combinations will be increased by 3; for the additional letter d may be combined with each of the former three, thereby forming three more combinations, the whole number being expressed by $1+2+3$:

thus ab, ac, bc, ad, bd, cd.

And by reasoning in the same manner, it will appear that the whole number of combinations of 2 in 5 quantities will be $1+2+3+4$: and in 6 quantities $1+2+3+4+5$, &c.

Therefore, if n be the number of things, the whole number of combinations, taken two by two, will be the series $1+2+3+4$, &c. continued to $n-1$ terms.

Now $1+2+3+4$, &c. is the 2*d*. order of figurate numbers (141), and the sum when continued to n terms is $\frac{n}{1}.\frac{n+1}{2}$ (144), therefore substituting $n-1$ for n gives $\frac{n-1}{1}\times\frac{n}{2}$ or $\frac{n}{1}\times\frac{n-1}{2}$ the sum of $1+2+3+4$, &c. continued to $n-1$ terms.

Let us now suppose the number of quantities in each combination *to be three.* Then if the quantities are only three (a, b, c)

there can be but 1 combination *abc*: But if the quantities are four, *a*, *b*, *c*, *d*, the number of combinations will be increased by 3; for *d* may be combined with *ab*, *ac*, *bc*, the combinations of two in the preceding letters *a*, *b*, *c*; therefore the whole number of combinations of 3 in 4 things will be expressed by $1+3$:

thus *abc*
abd, *acd*, *bcd*.

And if the quantities are augmented to five *a*, *b*, *c*, *d*, *f*, the combinations will be increased by 6 (or $1+2+3$) the combinations of 2 in the 4 letters *a*, *b*, *c*, *d*; for *f* may be combined with every two of them; therefore the combinations in this case is denoted by $1+3+6$:

thus *abc*
abd, *acd*, *bcd*
abf, *acf*, *bcf*, *adf*, *bdf*, *cdf*.

It therefore appears that the combinations of 3 in 6 things will be $1+3+6+10$; in seven $1+3+6+10+15$; and in n quantities $1+3+6+10+15$, &c., continued to $n-2$ terms; which series is the 3*d*. order of figurate numbers (141); whence, by substituting $n-2$ for n in the general expression $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}$ (144) we have $\frac{n-2}{1}\times\frac{n-1}{2}\times\frac{n}{3}$, or $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$ for the combinations of 3 in n quantities.

And if the number of quantities in each combination be 4, we shall get the 4*th*. order of figurate numbers, or $1+4+10+20$, &c. continued to $n-3$ terms for the combinations in n quantities; whence, by putting $n-3$ for n in the same general expression (144) the result is $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ for the number of combinations in that case.

Hence the combinations of two things in n things is $\frac{n}{1} \times \frac{n-1}{2}$.

$$\text{of three} \ldots\ldots\ldots \quad \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}.$$

$$\text{of four} \ldots\ldots \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}.$$

&c. &c.

Therefore *universally*, if m be the number of things in each combination, then $\frac{n}{1} . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} . \frac{n-4}{5}$, &c. continued to m factors, will give the whole number of combinations.

Examples.

1. How many combinations of 4 letters in the 24?

Here $n = 24$, and $m = 4$:

Therefore $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} = \frac{n-3}{4} = \frac{24 \times 23 \times 22 \times 21}{1 \times 2 \times 3 \times 4} = 23 \times 22 \times 21 = 10626$ the *answer*.

2. How many different hands can be held at the game of cribbage, if 6 cards is the deal?

Here $n = 52$, and $m = 5$.

And $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \frac{n-4}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{1 \times 2 \times 3 \times 4 \times 5} = 26 \times 17 \times 10 \times 49 \times 12 = 2598960$ the *answer*.

3. An old captain, who had often been successful in war, on being asked what reward he expected for his past services, desired a farthing only for every different file of 6 men he could make with his company which consisted of 100 men: what is the amount of his request?

$\frac{100 \times 99 \times 98 \times 97 \times 96 \times 95}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$, $= 1192052400$ the number of files or farthings, equal to 1241721*l.* 5*s.* the *answer*.

163. Beside the preceding, there are other kinds of combinations, as the *composition of quantities*, or when a given number of things are to be taken or combined from several setts, &c. The different cases however, are too numerous to be brought under any general rule.

We shall add a few miscellaneous examples with the methods of solution.

1. Suppose 4 ranks of men, 9 men in each rank; now how many ways can 4 men be chosen, 1 man being taken from every rank?

Since each man in one rank can be chosen with each man in another rank, the number of *twos* that can be formed out of two ranks will be 9 times 9 or 81: and because each man in a third rank can be taken, (or combined) with each of the 81 *twos*, the number of *threes* that can be chosen from three ranks is 81×9 or 729: again, each in the *4th.* rank can be combined with each of the 729 *threes*; therefore 729×9 or $9^4 = 6561$ is the number of compositions, or the answer.

And if the ranks (or setts) are unequal, the number of compositions will be found exactly in the same manner; *ex. gr.* suppose 5, 6, 8, and 9, are in the respective ranks, then $5 \times 6 \times 8 \times 9$ (instead of $9 \times 9 \times 9 \times 9$ will be the number of compositions.

2. How many changes or chances are there in throwing 4 dice?

If we suppose 4 ranks, 6 in each rank, and each combination to be 1 from every rank, then, as in the preceding example, $6 \times 6 \times 6 \times 6$ or $6^4 = 1296$ is the number of different throws or chances.

3. Let there be 3 setts of different things, 4 in each sett, to find the compositions of 4, supposing 1, or more, is taken from each sett every time?

The combinations of 2 in 4 are 6, and since each in one sett can be combined with the twos in another, the compositions of the twos in one sett with the ones in another are 6×4 or 24, therefore the whole number of compositions of 3 in 2 setts is $24 \times 2 = 48$:

Again, each single one in the 3*d.* sett can be combined with each of the 48 threes in the other two, making 48×4 compositions; and as the combinations of 2 setts in 3 are 3, consequently $48 \times 4 \times 3$ or $576 = 12^2 \times 4$, *viz.* the square of the number of things multiplied by the number in each composition, is the *answer.*

4. How many changes can be rung with 4 bells out of 8?

The combinations of 4 in 8 are $\frac{8.7.6.5}{1.2.3.4} = 70$, which multiplied by $1 \times 2 \times 3 \times 4$, the changes in 4, make 1680 the *answer.*

5. How many different numbers can be made out of an unit, 2 twos, 3 threes, and 4 fours, taking four figures at a time?

To solve this problem it may be necessary to consider the changes or alternations that can take place in a form of this kind *aaabbc* where there are several things of one sort, and several of another.

If there are three things *aac*, two of them being alike, then *aac*, *aca*, *caa* are their variations; but when all are different, as *a*, *b*, *c*, the permutations will be $1 \times 2 \times 3$ which is 1×2 (the changes in 2 things) times greater than the changes in *aac*, the variations in *aac* are therefore expressed by $\frac{1 \times 2 \times 3}{1 \times 2}$.

And if *dddf* are 4 things where three are alike, all the variations are *dddf*, *ddfd*, *dfdd*, *fddd*, or $1 \times 2 \times 3$ (the changes in 3 things) times less than $1 \times 2 \times 3 \times 4$ the permutations when all four are different, consequently $\frac{1 \times 2 \times 3 \times 4}{1 \times 2 \times 3}$ will denote the variations in *dddf*; and if these forms are combined, it follows that the variations in *aacdddf* will be truly expressed by $\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{(1 \times 2)\ (1 \times 2 \times 3)}$, where the numerator is the permutations in 7 things (the number of letters), the denominator being the product of the respective changes in 2, and 3 things, the repetitions of *a* and *d*.

That any number of like things standing next to one another do not admit of a variation, is manifest from a repetition of any of the numeral digits; thus, the number 333 is not changed by any shifting of its figures.

Now let the proposed figures 1223334444 be represented by *abbcccdddd*; then combining them by fours, we get the following forms:

d^4	variations
d^3c, d^3b, d^3a, c^3d, c^3b, c^3a	4
d^2c^2, d^2b^2, b^2c^2	6
d^2bc, d^2ba, d^2ac, c^2db, c^2ba, c^2ad, b^2dc, b^2ca, b^2ad	12
$dcba$	24

The variations in d^3c, or d^3b, &c. are 4, in d^2bc, &c. 12;

$$\text{therefore}\quad \left.\begin{array}{r} 4\times6=24\\ 6\times3=18\\ 12\times9=108\\ 24\times1=24\\ d^4\ldots\ldots\ldots\ 1\end{array}\right\}\text{the variations multiplied by the number of combinations.}$$

sum 175 the *answer*, or all the combinations of 4 figures with their variations.

6 How many different numbers can be made with the same figures as in the preceding example (1223334444) supposing all the figures to be in every number?

By the last problem, $\frac{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}{(1\times2)\ (1\times2\times3)\ (1\times2\times3\times4)}=12600$ the *answer*.

7. To find all the compositions or different integral numbers that can be formed by means of the nine digits, taking them by twos, by threes, &c. up to nines.

This is the same thing as finding the whole number of compositions in 9 ranks of the 9 digits when combined by 2, by 3, by 4, &c. up to 9 at a time:

Therefore, by the first example in this article, the compositions of 2 in two ranks 9 in each rank is 9×9 or 9^2, of 3 in three ranks is 9^3, of 4 in four ranks 9^4, &c.

Hence, if n be any number of things or quantities, the sum of all the possible compositions by twos, by threes, &c. up to n's, will be the sum $n+n^2+n^3+\ldots\ldots\ldots\ n^n$, which is a geometrical progression having n for the first term, for the ratio, and also for the number of terms; and the sum will be $n\times\frac{n^n-1}{n-1}$, (153), or, in the present case, $9\times\frac{9^9-1}{2-1}=435848049$ the answer; being the number of different integers in which there is no cypher, from 1 to 999999999 both inclusive.

The doctrine of permutations, combinations, &c. is of considerable use in several parts of the mathematics; particularly in the calculation of annuities and chances.

Of NEWTON'S BINOMIAL THEOREM.

164. This is called the Binomial Theorem on account of its being a general formula for readily obtaining the powers, or roots, of any expression consisting of two terms. The method of denoting the coefficients admits of some variation; but one of the most commodious forms is the following:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n}{1}.\frac{n-1}{2}a^{n-2}b^2 + \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 + \&c.$$

If $(a-b)^n$ is the binomial and n is a positive integer, the 2*d*, 4*th*, 6*th*, &c. terms are negative (102).

In this theorem the index n may be any number, whole or fractional, positive or negative, and herein consists its principal excellence; because if n is a proper fraction, we obtain an approximating series for the root of the binomial denoted by that fractional exponent. A few examples will be sufficient to point out the method of substitution.

1. To find the cube or 3*d*. power of $a+b$.

Here $n=3$, or $(a+b)^n = (a+b)^3$.

And $n=3$ the coefficient of the 2*d*. term.

$\frac{n}{1}.\frac{n-1}{2} = \frac{3}{1} \times \frac{3-1}{2} = 3$ the coefficient of the 3*d*. term.

$\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3} = \frac{6}{6} = 1$ the coefficient of the 4*th*. term.

Therefore $a^3 + 3a^{3-1}b + {}^{3-2}b^2 + {}^{3-3}b^3$ or $a^3 + 3a^2b + 3ab^2 + b^3$, ($a^{3-3}$ being 1) is the required cube.

2. To find an approximating series for the square root of x^2+1. (107 *Examp*. 3).

The expression is $(x^2+1)^n$ or $x^2+1)^{\frac{1}{2}}$, where $x^2=a$, $1=b$, and $n=\frac{1}{2}$.

$a^n=(x^2)^{\frac{1}{2}}=x$ the first term.

$na^{n-1}b=\frac{1}{2}\left(x^2\right)^{\frac{1}{2}-1}=\frac{1}{2}\left(x^2\right)^{-\frac{1}{2}}=\frac{1}{2}x^{-1}=\frac{1}{2x}$ the 2*d*. term.

$\frac{n}{1}\times\frac{n-1}{2}a^{n-2}b^2=-\frac{1}{8}\left(x^2\right)^{\frac{1}{2}-2}=-\frac{1}{8}x^{-3}=-\frac{1}{8x^3}$ the 3*d*. term.

$\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}b^3=\frac{1}{16}\left(x^2\right)^{-2\frac{1}{2}}=\frac{1}{16}x^{-5}=\frac{1}{16x^5}$ the 4*th* term.

$\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}a^{n-4}b^4=-\frac{5}{128}\left(x^2\right)^{-3\frac{1}{2}}=-\frac{5}{128x^7}$ the 5*th*. term, &c.

Hence $x+\frac{1}{2x}-\frac{1}{8x^3}+\frac{1}{16x^5}-\frac{5}{128x^7}+$ &c. is the series required.

3. Let it be required to convert $\frac{1}{(a+b)^2}$ or $(a+b)^{-2}$ into a series.

Here $n=-2$. And a^{-2} or $\frac{1}{a^2}$ is the first term.

$na^{n-1}b=-2a^{-3}b=-\frac{2b}{a^3}$ the second.

$\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2=+3a^{-4}b^2=+\frac{3b^2}{a^4}$ the third.

$\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3=-4a^{-5}b^3=-\frac{4b^3}{a^5}$ the fourth, &c.

Therefore $\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+$ &c. where the law of continuation is manifest.

4. To expand $\frac{1}{(a+b)^{\frac{1}{2}}}$ or $(a+b)^{-\frac{1}{2}}$ into a series.

In this example $n=-\frac{1}{2}$.

And $a^n=a^{-\frac{1}{2}}=\frac{1}{a^{\frac{1}{2}}}$ the first term.

$$na^{n-1}b = -\tfrac{1}{2}a^{\frac{3}{2}}b = -\frac{b}{2a^{\frac{3}{2}}} \text{ the second.}$$

$$\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2 = +\frac{3}{8}a^{-\frac{5}{2}}b^2 = +\frac{3b^2}{8a^{\frac{5}{2}}} \text{ the third.}$$

$$\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3 = -\frac{5}{16}a^{-\frac{7}{2}}b^3 = -\frac{5b^3}{16a^{\frac{7}{2}}} \text{ the fourth.}$$

whence $$\frac{1}{(a+b)^{\frac{1}{2}}} = \frac{1}{a^{\frac{1}{2}}} - \frac{b}{2a^{\frac{3}{2}}} + \frac{3b^2}{8a^{\frac{5}{2}}} - \frac{5b^3}{16a^{\frac{7}{2}}} + \&c.$$

5. A trinomial may be raised to any given power by considering the sum or difference of two terms as one factor:

Thus $$(a+b+c)^n = \left(a+\overline{b+c}\right)^n = a^n + na^{n-1}(b+c) + \frac{n}{1}\cdot\frac{n-1}{2}$$
$$a^{n-2}(b+c)^2 + \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}(b+c)^3 + \&c.$$

Demoivre, *Miscell. Analyt.* and Dr. Cheyne, *Method. Inv.* have extended the rule to any multinomial.

165. Among the different investigations that have been given of the preceding theorem, the following, by means of the continued product of binomial factors, seems the most natural and easy when the exponent is a whole number.

If $a+b$, $a+c$, $a+d$, &c. are a series of binomial factors, to determine the coefficients of a in the product $(a-b)$ $(a+c)$ $(a+d)$ &c.

By actual multiplication we get

$$(a+b)(a+c) = a^2 + \left.\begin{matrix} b \\ c \end{matrix}\right\} a + bc$$

$$(a+b)(a+c)(a+d) = a^3 + \left.\begin{matrix} b \\ c \\ d \end{matrix}\right\} a^2 + \left.\begin{matrix} bc \\ bd \\ cd \end{matrix}\right\} a + bcd$$

$$(a+b)(a+c)(a+d)(a+f) = a^4 + \left.\begin{matrix} b \\ c \\ d \\ f \end{matrix}\right\} a^3 + \left.\begin{matrix} bc \\ bd \\ bf \\ cd \\ cf \\ df \end{matrix}\right\} a^2 + \left.\begin{matrix} bcd \\ bcf \\ bdf \\ cdf \end{matrix}\right\} a + bcdf, \ \&c.$$

Hence it appears, that the coefficient of a in the 2*d.* term is always the sum of the other quantities b, c, d, &c.

The coefficient in the 3d. term the sum of all their products or combinations two by two:

The coefficient in the 4th, term the sum of all their products combined three by three, &c. &c.

Therefore, if n be the number of factors $a+b$, $a+c$, $a+d$, &c. the number of letters in the coefficient of the 2d. term will also be denoted by n:

The number of their combinations two by two in the 3d. term, by $\frac{n}{1}.\frac{n-1}{2}$, (161).

The combinations three by three in the 4th. by $\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}$, &c.

Now suppose c, d, f, &c. to be each equal to b;

then $(a+b)\ (a+c)\ (a+d)\ (a+f)$ &c. $=(a+b)\ (a+b)\ (a+b)\ (a+b)$, &c. $=(a+b)^n$;

$$\text{that is, } (a+b)^4=a^4+\left.\begin{matrix}b\\b\\b\\b\end{matrix}\right\}a^3+\left.\begin{matrix}b^2\\b^2\\b^2\\b^2\\b^2\\b^2\end{matrix}\right]a^2+\left.\begin{matrix}b^3\\b^3\\b^3\\b^3\end{matrix}\right\}a+b^4, \text{ when } n=4.$$

It follows then, (n being the number of factors) that the coefficient of a in the 2d. term is nb:

in the 3d. $\frac{n}{1}.\frac{n-1}{2}b^2$:

in the 4th. $\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}b^3$:

in the 5th. $\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}b^4$, &c.

And since a^n, a^{n-1}, a^{n-2}, a^{n-3}, &c. are the successive powers of a, we have $(a+b)^n=a^n+nba^{n-1}+\frac{n}{1}.\frac{n-1}{2}b^2a^{n-2}+\frac{n}{1}.\frac{n-1}{1}.\frac{n-2}{3}b^3a^{n-3}+$ &c. where it is evident the series will terminate at $n+1$ terms, as in the first example.

166. But the preceding method of deriving the law of the coefficients has not been considered as sufficiently general, because the value of the index n is restricted to a positive integer. The following investigation however, by means of a trinomial $(1+a+b)^n$ is not liable to the same objection *. In this

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* Manning's *Algebra*, vol. 2.

process we consider the trinomial as consisting of two terms only, $(\overline{1+a}+b)^n$, and $(1+\overline{a+b})^n$, then as these quantities are the same, the results from similar operations must be equal.

It is manifest from multiplication and the extraction of roots, that when a binomial $a+b$ is raised to the nth. power, the two first terms will always be $a^n + na^{n-1}b$, *viz.* $(a+b)^n = a^n + na^{n-1}b$, &c.

Thus if $n=2$, then $(a+b)^2 = a^2 + 2a^{2-1}b$, &c. $= a^2 + 2ab$, &c.

$n=5$........ $(a+b)^5 = a^5 + 5a^{5-1}b$, &c. $= a^5 + 5a^4b$, &c.

And when n is negative, or a fraction, the two leading terms are found exactly in the same manner:

Thus, let $n=-2$;

Then $(a+b)^{-2} = a^{-2} + (-2a^{-2-1}b)$ &c. $= \frac{1}{a^2} - \frac{2b}{a^3}$ &c. the result by actual division; for $(a+b)^{-2} = \frac{1}{(a+b)^2} = \frac{1}{a^2} - \frac{2b}{a^3}$ &c.

Again, let $n=\frac{1}{2}$:

Then $(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2}a^{\frac{1}{2}-1}b$ &c. $= a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}}$ &c. the terms which we get by extracting the square root of $a+b$. And other proofs are obtained in a similar manner with different values of n.

Hence, if the theorem is put in this form,

$$(a+b)^n = a^n + na^{n-1}b + Pa^{n-2}b^2 + Ra^{n-3}b^3 + Sa^{n-4}b^4 \text{ \&c.}$$

the thing to be determined is the law of the coefficients P, R, &c. independent of any numeral value of n: P, R, &c. being the coefficients of the 3*d*, 4*th*, &c. terms of the powers of any binomial.

Let $1+a$ be considered as one term;

$$\text{Then } (\overline{1+a}+b)^n = (1+a)^n + n(1+a)^{n-1}b, \text{ \&c.}$$

$$= (1+a)^n = 1 + na + Pa^2 + Ra^3 + Sa^4 + \text{\&c.}$$

$$+ (1+a)^{n-1} \times nb, \text{ \&c.}$$

Now, suppose p, r, &c. to denote the coefficients of the 3*d*, 4*th*, &c. terms of $(1+a)^{n-1}$ when expanded.

Then 1^{n-1}, or 1 is the first term;

$(n-1)\,1^{n-1}a$, or $a\,(n-1)$ the *2d.*

pa^2 the *3d.*

ra^3 the *4th*, &c.

$$\text{Therefore } (1+a)^{n-1}\times nb = (1+a\,(n-1)+pa^2+ra^3 \text{ \&c.})\,nb$$
$$= nb+n\,(n-1)\,ab+npa^2b+nra^3b \text{ \&c.}$$

$$\text{And } (\overline{1+a}+b)^n = \left\{\begin{array}{l} 1+na+Pa^2+Ra^3+Sa^4, \text{ \&c.} \\ +nb+n\,(n-1)\,ab+npa^2b+nra^3b, \text{ \&c.}\end{array}\right.$$

Again, let $a+b$ be taken as a single term;

Then $(1+\overline{a+b})^n = 1+n\,(a+b)+P\,(a+b)^2+R\,(a+b)^3+S(a+b)^4$, &c. now, to compare this with the preceding series, the terms of $(a+b)^2$, $(a+b)^3$, &c. which involve the *2d, 3d,* &c. powers of b, may be omitted: and we shall have

$$\begin{aligned}(1+\overline{a+b})^n = 1+na+nb+Pa^2+Pab&\\ +Ra^3+3ra^2b&\\ +Sa^4+4Sa^3b&, \text{ \&c.}\end{aligned}$$

$$= 1+na+Pa^2+Ra^3+Sa^4, \text{ \&c.}$$
$$+\,nb+2Pab+3ra^2b+4Sa^3b, \text{ \&c.}$$ this result must be equal to that obtained from $(\overline{1+a}+b)^n$, whence by omitting $1+na+Pa^2$, &c. which is common to both, we get

$$nb+n\,(n-1)\,ab+npa^2b+nra^3b, \text{\&c.} = nb+2Pab+3ra^2b+4Sa^3b, \text{ \&c.}$$

Now the powers of a and b on one side of the equation being the same as on the other, their respective coefficients must be equal;

$$\begin{array}{lll} \textit{viz,} & n=n & np=3R \\ & n\,(n-1)=2P & nr=4S, \text{ \&c.}\end{array}$$

whence $P=\frac{n\,(n-1)}{2}$, that is, the coefficient of the *3d.* term of $(1+a)^n$ is half the product of the index and the index *minus* 1; hence p or the coefficient of the *3d.* term of $(1+a)^{n-1}$ will be $\frac{(n-1)\,.\,(n-2)}{2}$, or half the product of the index and the index *minus* 1.

Now $\frac{n\,(n-1)\,.\,(n-2)}{2}=3R$, whence $R=\frac{n\,(n-1)\,.\,(n-2)}{2\,.\,3}$ the coefficient of the *4th.* term: and $r=\frac{(n-1)\,.\,(n-2)\,.\,(n-3)}{2\,.\,3}$; and since $nr=4S$, S will be $=\frac{n\,(n-1)\,.\,(n-2)\,.\,(n-3)}{2\,.\,3\,.\,4}$ the coefficient of the *5th.* term, &c.

This celebrated Theorem has been demonstrated various ways; but sometimes from principles which may be said to depend on the theorem itself. Newton seems to have found the law of the coefficients by *induction:* a method which has led to the most important discoveries in science.

OF CONTINUED FRACTIONS.

167. CONTINUED Fractions have an integer and a fraction for the denominator, and the fraction in that denominator has also an integer and a fraction for its denominator; in like manner, the denominator of the last fraction is composed of an integer and a fraction, and so on. These fractions are generated by division after the manner of reducing a fraction to its lowest terms in Arithmetic:

Thus, to reduce $\frac{761}{2385}$ to a continued fraction, let both terms of the fraction be divided by 761

and we have $\frac{761}{2385} = \cfrac{1}{3\frac{102}{761}}$

Again, if both terms of the fraction $\frac{102}{761}$ are divided by the numerator 102

then $\cfrac{1}{3\frac{102}{761}} = \cfrac{1}{3+\cfrac{1}{7+\cfrac{47}{102}}}$

And both terms of the fraction $\frac{47}{102}$ divided by 47

gives $\cfrac{1}{3+\cfrac{1}{7+\cfrac{47}{102}}} = \cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{8}{47}}}}$

The next reduction is performed by dividing both terms of the fraction $\frac{8}{47}$ by 8,

$$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{8}{47}}}} = \cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\cfrac{7}{8}}}}}$$

Again, divide both terms of $\frac{7}{8}$ by 7

$$\text{and we have } \cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\frac{7}{8}}}}} = \cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1\frac{1}{7}}}}}}$$

$$\text{Therefore } \frac{761}{2358} = \cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1+\frac{1}{7}}}}}} \quad \text{is the continued fraction.}$$

The method of deriving the primitive fraction $\frac{761}{2385}$ by a reverse operation commencing at the last fraction $\frac{1}{7}$ is obvious from the preceding operation. But as one principal use of these continued fractions is to find two numbers that shall be nearly in the same proportion as two given numbers, but consisting of fewer figures, we must begin with the first fraction $\frac{1}{3\ \&c.}$ to shew the order in which such numbers are found.

168. Suppose then, 2385 and 761 are two given numbers, and that we would find two other numbers that shall exhibit their ratio nearly in fewer figures:

Now by means of the continued fraction we get 5 comparisons of such numbers, or 5 fractions approaching nearer and nearer to the given ratio or fraction $\frac{761}{2385}$,

Thus

$\frac{1}{3} = \frac{1}{3}$ the first, which is greater than $\frac{761}{2385}$;

$\cfrac{1}{3+\cfrac{1}{7}} \cdots = \frac{7}{22}$ the 2*d.* too little, or less than $\frac{761}{2385}$;

$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2}}} \cdots = \frac{15}{47}$ the 3*d.* too great, but nearer than the last.

$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5}}}} \cdots\cdots = \frac{82}{257}$ the 4*th.* too little.

$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1}}}}} \cdots\cdots = \frac{97}{304}$ the 5*th.* too great, but nearer $\frac{761}{2385}$ than either of the preceding.

$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{7}}}}}} \cdots\cdots = \frac{761}{2385}$ the fraction proposed.

Here we may observe that the approximations are alternately too great and too little, but their values become nearer to the value of the given fraction as the numerators and denominators increase. We shall now give the method of obtaining all the other approximations or expressions by means of the two first, and the quotients that follow.

169. Let *a, b, c, d, f,* &c. be the quotients found in succession by reducing any fraction $\frac{n}{m}$ to a continued fraction.

Then $\frac{n}{m} = \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{f+1 \text{ \&c.}}}}}}$

And $\frac{1}{a}$ is the first expression or approximation:

$$\cfrac{1}{a+\cfrac{1}{b}} \cdots\cdots = \frac{b}{ab+1} \text{ the 2d.}$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}} \cdots\cdots = \frac{bc+1}{abc+a+c} \text{ the 3d.}$$

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}} \cdots\cdots = \frac{bcd+b+d}{abcd+ab+ad+cd+1} \text{ the 4th, \&c.}$$

Hence it appears that the 3*d.* expression is found by multiplying the numerator and denominator of the 2*d.* by the third quotient c, and adding 1 the numerator of the first expression $\left(\frac{1}{a}\right)$ to the first of these products, and the denominator a to the second:

And, if the terms of the 3*d.* expression are multiplied by the fourth quotient d, and the products augmented in the same manner by the terms of the 2*d.* the result is the 4*th.* expression; and so on:

Thus

$$\frac{c \times b+1}{c \times (ab+1)+a} = \frac{bc+1}{abc+a+c} \text{ the 3d. expression}$$

$$\frac{d\,(bc+1)+b}{d\,(abc+a+c)+ab+1} = \frac{bcd+b+d}{abcd+ab+ad+cd+1} \text{ the 4th.}$$

$$\frac{f\,(bcd+b+d)+bc+1}{f\,(abcd+ab+ad+cd+1)+abc+a+c} \text{ the 5th, \&c.}$$

170. These expressions are convenient for finding the vulgar fraction answering to a recurring decimal:

Thus, suppose the decimal .76923 &c.

$$\text{Then } \frac{76923}{100000} = \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{3\frac{1}{7692}}}}$$

Here the three quotients are $1=a$, $3=b$, $3=c$, these being substituted in the $3d.$ expression, give $\frac{9+1}{9+1+3}=\frac{10}{13}$ the required fraction.

Remark. It is not necessary to keep exactly to the preceding form in reducing a given fraction to a continued one. The process of division may be set down as it is in finding the greatest common measure, and the continued fraction formed afterwards by means of the quotients, as in the following example:

171. To find the ratio of the diameter of a circle to its circumference, nearly, in small integer numbers.

If the diameter is 1, the circumference will be 3.141593 nearly (*Art.* 266. Mensuration). Hence the given fraction is $\frac{1000000}{3141593}$.

$$\begin{array}{llll}
1000000)\ 3141593\ (3 & & & \\
\ \underline{3000000} & & & \\
\ \ 141593) & 1000000\ (7 & & \\
 & \ \underline{991151} & & \\
 & \ \ \ 8849) & 141593\ (16\ \text{nearly} & \\
 & & \ \underline{8849} & \\
 & & \ 53103 & \\
 & & \ \underline{53094} &
\end{array}$$

Therefore $\frac{1000000}{3141593}=\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{16}}}$

The quotients are $3=a$, $7=b$, and $16=c$; and if the two first are substituted in the $2d.$ expression, we have $\frac{7}{21+1}=\frac{7}{22}$ the ratio that Archimedes assigned, which is used for common purposes.

By taking in the quotient c, the $3d.$ expression gives $\frac{7\times16+1}{3\times7\times16+3+16}$ $=\frac{113}{355}$ an approximation nearer the truth than the fraction $\frac{1000000}{3141593}$.

The idea of these fractions seems to have originated with Lord Brouncker, the first President of the Royal Society. Afterwards Huygens extended their application: and since that time Lord Stanhope, Euler, and particularly Legrange, have greatly improved the theory, and shown their use in the extraction of roots, the summation of series, &c.

OF RECURRING SERIES.

172. **RECURRING** series are so constituted that each term has a constant relation to some given number of the preceding terms taken always in the same order:

thus if $1 + 2x + 3x^2 + 4x^3 + 5x^4 +$ &c. be the series,

then the 4th. term (for example) is $= 2x \times$ 3d. term $- x^2 \times$ 2d. term:

The 5th. term $= 2x \times$ 4th. term $- x^2 \times$ 3d. term, and so on.

And the expression $2x - x^2$ is called the *scale of relation* of the terms; this scale however, is sometimes exhibited by the coefficients only, (as $2 - 1$ the coefficients of $2x - x^2$).

Again, suppose $1 + 3x + 5x^2$ are the three first terms of a series, and assuming $3 - 2 + 4$ for the scale of the coefficients, or $3x - 2x^2 + 4x^3$ for the scale of relation of the terms;

Then $3x \times$ ($5x^2$ the 3d. term) $- 2x^2 \times$ ($3x^2$ the 2d. term) $+ 4x^3 \times$ (1 the 1st. term) $= 13x^3$ the 4th. term of the series:

Also $3x \times 13x^3 - 2x^2 \times 5x^2 + 4x^3 \times 3x = 41x^4$ the 5th. term of the series, &c.

And the series is $1 + 3x + 5x^2 + 13x^3 + 41x^4 + 117x^5$ &c.

173. Let $\overset{A}{a} + \overset{B}{bx} + \overset{C}{cx^2} + \overset{D}{dx^3} + \overset{E}{fx^4} +$ &c. be a series, and $tx - sx^2$ the scale of relation of the terms;

$$\begin{aligned} \text{Then } cx^2 &= tx \times bx - sx^2 \times a = tbx^2 - sax^2 \\ dx^3 &= tx \times cx^2 - sx^2 \times bx = tcx^3 - sbx^3 \\ fx^4 &= tx \times dx^3 - sx^2 \times cx^2 = tdx^4 - scx^4 \\ &\text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.} \end{aligned}$$

whence it is evident that all the terms after the two first, may be exhibited by means of those two terms and the scale of relation.

Now suppose it is required to find the sum of the above series *infinitely continued:*

Let A, B, C, &c. denote the terms a, bx, cx^2, &c. respectively:

$$\begin{aligned}\text{Then } A &= A\\ B &= B\\ C &= Btx - Asx^2\\ D &= Ctx - Bsx^2\\ E &= Dtx - Csx^2\\ \text{\&c.} & \quad \text{\&c.}\end{aligned}$$

here it is manifest that the sum $A + B + C + D + E$ (the first column) is equal to both the other columns added together:

Put $A + B + C +$ &c. $= S$; then $B + C + D$ &c. $= S - A$:

The sum of the terms in the 2*d*. column is $= A + B + (B + C + D \text{ \&c.})\, tx$;

but $B + C + D$, &c. is $= S - A$,

therefore the 2*d*. column $= A + B + (S - A)\, tx = A + B + Stx - Atx$:

And since the series is supposed to be infinite,

$A + B + C +$ &c. in the 3*d*. column will be $= S$,

therefore the 3*d*. column, or $-(A + B + C)\, sx^2$ is $= -Ssx^2$;

And $A + B + Stx - Atx - Ssx^2$ is the aggregate of the second and 3*d*. columns, which sum is equal to the first,

or $A + B + Stx - Atx - Ssx^2 = S$,

whence $\dfrac{A + B - Atx}{1 - tx + sx^2} = S$ the sum of the series infinitely continued.

It is manifest this expression will not give the sum of any proposed number of terms, because if that number be *less* than infinite, the number of terms in the 3*d*. column will *always* be less by *two* than those in the first or second columns, and consequently $A + B + C +$ &c. in the 3*d*. column cannot in *that case* be $= S$. And since it is impossible to find the sum of an infinite number of any given quantities, it follows that $a + bx + cx^2 +$ &c. must be a converging series, that is, a series where the terms constantly diminish or approach to the limit o, which may be considered as the least or last term.

If $a=1$, $b=2$, $t=2$, and $s=1$, the series $a+bx+cx^2+$ &c. becomes $1+2x+3x^2+$ &c.

$$\text{And } \frac{A+B-Atx}{1-tx+sx^2}=\frac{1+2x-2x}{1-2x+x^2}=\frac{1}{1-2x+x^2}=\frac{1}{(1-x)^2}=S.$$

Now $\frac{1}{(1-x)^2}$ expanded by division, gives the proposed series,

$$\begin{array}{lllll}
\text{Thus } 1-2x+x^2)\ 1 & & & & (1+2x+3x^2+4x^3+\text{\&c.} \\
\underline{1-2x+x^2} & & & & \\
\quad +2x-x^2 & & & & \\
\quad \underline{+2x-4x^2+2x^3} & & & & \\
\qquad\quad +3x^2-2x^3 & & & & \\
\qquad\quad \underline{+3x^2-6x^3+3x^4} & & & & \\
\qquad\qquad\quad +4x^3-3x^4 & & & & \\
\qquad\qquad\quad \underline{+4x^3-8x^4+4x^5} & & & & \\
\qquad\qquad\qquad\quad +5x^4-4x^5 & & & & \\
\qquad\qquad\qquad\qquad\quad \text{\&c.} & & & &
\end{array}$$

It therefore appears, that summing a recurring series is only discovering the *radix* or fraction from which it was, or might be derived, (72):

Thus, suppose the series to be $1+3x+4x^2+7x^3+11x^4+$ &c. where the scale of relation of the coefficients is $1+1$, *viz.* the coefficient of any term (after the two first) is the sum of the two preceding coefficients. Then $t=1$, and $s=1$, and the latter being positive, the expression $\frac{A+B-Atx}{1-tx+sx^2}$ becomes $\frac{1+2x}{1-x-x^2}=S$, the sum of the series infinitely continued. For $\frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+$ &c. by actual division.

174. In order to find the sum of any number (n) of terms of the series $1+2x+3x^2+4x^3+$ &c. (for example), it is to be observed that the nth term is nx^{n-1}, and consequently the terms which follow will be $(n+1)x^n+(n+2)\,x^{n+1}+(n+3)\,x^{n+2}+$ &c. where the scale of relation of the coefficients is $2-1$ as before: therefore substituting $(n+1)x^n$ and $(n+2)\,x^{n+1}$, re-

spectively, for A and B in the expression $\frac{A+B-Atx}{1-tx+sx^2}$, and we have

$$\frac{(n+1)\,x^n+(n+2)\,x^{n+1}-(n+1)\,x^n tx}{1-tx+sx^2}=\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}$$ the

sum of the series $(n+1)\,x^n+(n+2)\,x^{n+1}+$ &c. *ad infinitum:*

Now it is evident that the difference of the expressions for the two sums will be the expression for the sum of n terms of the series,

that is $$\frac{1}{(1-x)^2}-\frac{(n+1)\,x^n-nx^{n+1}}{(1-x)^2}=\frac{1-(n-1)\,x^n+nx^{n+1}}{(1-x)^2}$$

which is the expression for the sum of n terms of the series $1+2x+3x^2+$ &c.

175. If the scale of relation of the coefficients consists of three terms $t+s+v$; then, proceeding as in *Art.* 173, we get the following expression for the sum of $a+bx+cx^2+$ &c. *in infin.*

viz. $$\frac{A+B+C-(A+B)\,tx-Asx^2}{1-tx-sx^2-vx^3}=S.$$

Hence it appears that when the scale consists of n terms, the expression will be

$$\frac{A+B+C+\ldots n-(A+B+\ldots\overline{n-1})\,tx-(A+\ldots\overline{n-2})\,sx^2\,\&c.}{1-tx-sx^2-vx^3-\ldots.(n+1)}=S;$$

Which is a general Theorem for the sum of an infinite recurring series; and from this the sum of any given number of the terms may be found as in the preceding article. But as the signs in the scale of relation are here supposed to be positive, care must be taken when negative signs occur, to make the substitution accordingly.

OF THE DIFFERENTIAL METHOD.

176. THIS is a method of summing series, &c. by means of the successive differences of their terms.

Let 1 7 28 84 210 462 924 1716, &c. be a series of numbers; Then taking the difference of the first and second, of the second and third, of the third and fourth, &c. and again the differences of those differences, and so on, we shall have the following orders of differences:

	1	7	28	84	210	462	924	1716 &c.
1st. order of differences	6	21	56	126	252	462	702	
2d. order	15	35	70	126	210	330		
3d. order	20	35	56	84	120			
4th. order	15	21	28	36				
5th. order	6	7	8					
6th. order	1	1						
	0							

Or suppose $a, b, c, d, f, g,$ &c. to be a series; then

1st. order of differences $b-a,\ c-b,\ d-c,\ f-d,\ g-f,$ &c.
2d. order .... $c-2b+a,\ d-2c+b,\ f-2d+c,\ g-2f+d,$ &c.
3d. order .... $d-3c+3b-a,\ f-3d+3c-b,\ g-3f+3d-c,$ &c.
4th. order .... $f-4d+6c-4b+a,\ g-4f+6d-4c+b,$ &c.
5th. order .... $g-5f+10d-10c+5b-a,$ &c.

177. Let $D^{\mathrm{I}}, D^{\mathrm{II}}, D^{\mathrm{III}}, D^{\mathrm{IV}}, D^{\mathrm{V}},$ &c. denote the first terms of the several orders of differences, respectively,

that is, put $D^{\mathrm{I}} = b-a$

$$D^{\mathrm{II}} = c-2b+a$$
$$D^{\mathrm{III}} = d-3c+3b-a$$
$$D^{\mathrm{IV}} = f-4d+6c-4b+a$$
$$D^{\mathrm{V}} = g-5f+10d-10c+5b-a$$

&c. &c.

Then by transposition we get the values of b, c, d, &c,

$$\begin{aligned} b &= a + D^{I} \\ c &= 2b - a + D^{II} \\ d &= 3c - 3b + a + D^{III} \\ f &= 4d - 6c + 4b - a + D^{IV} \\ g &= 5f - 10d + 10c - 5b + a + D^{V} \\ &\text{\&c.} \qquad \qquad \text{\&c.} \end{aligned}$$

But $2b - a = b + (b - a) = a + D^{I} + D^{I} = a + 2D^{I}$, (because $b = a + D^{I}$ and $b - a = D^{I}$) therefore $c = a + 2D^{I} + D^{II}$.

Also, since $3b - a = 3\ (a + D^{I}) + a$;

we have $3c - 3b + a = 3\ (a + 2D^{I} + D^{II}) - 3\,(a + D^{I}) + a = a + 3D^{I} + 3D^{II}$,

whence $d = a + 3D^{I} + 3D^{II} + D^{III}$.

And in like manner we get

$$f = a + 4D^{I} + 6D^{II} + 4D^{III} + D^{IV}.$$

Hence

$$\begin{aligned} a &= a \\ b &= a + D^{I} \\ c &= a + 2D^{I} + D^{II} \\ \text{(A)} \qquad d &= a + 3D^{I} + 3D^{II} + D^{III} \\ f &= a + 4D^{I} + 6D^{II} + 4D^{III} + D^{IV} \\ g &= a + 5D^{I} + 10D^{II} + 10D^{III} + 5D^{IV} + D^{V} \\ &\text{\&c.} \qquad \qquad \text{\&c.} \end{aligned}$$

where the law of continuation is evident.

Hence it appears that the coefficients of a, D^{I}, D^{II}, D^{III}, &c. in the expression for the $(n+1)$*th.* term of the series a, b, c, &c. are the coefficients of a binomial raised to the nth. power; that is, the $(n+1)$ *th.* term is $a + nD^{I} + n.\frac{n-1}{2} D^{II} + n.\frac{n-1}{2}.\frac{n-2}{3} D^{III}$, &c. (164)

Thus, for example, if the number of terms be 5, or $n = 5$, the *6th.* or $(n+1)$ *th.* term is $a + 5D^{I} + 5.\frac{5-1}{2} D^{II} + 5.\frac{5-1}{2}.\frac{5-2}{3} D^{III}$ &c. or $a + 5D^{I} + 10D^{II}$ &c. the value of g the *6th.* term.

Therefore substituting $n+1$ for n, the nth. term of the series a, b, c, &c. will be

$$a + (n-1)\,D^{I} + \frac{n-1}{1}.\frac{n-2}{2} D^{II} + \frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3} D^{III} + \text{\&c.}$$

178. A general expression for the sum of any number (n) of the terms of the series, is readily obtained from the aggregate sum of the perpendicular columns as they stand in the expressions (A):

Thus, the coefficients in the columns $a, a, a,$ &c. $D^{I}, 2D^{I}, 3D^{II},$ &c. are the several orders of figurate numbers (141):

Now the sum of $a+a+a+$ &c. to n terms is na:

of $D^{I}+2D^{I}+3D^{I}+$ &c. to $n-1$ terms is $n.\frac{n-1}{2}D^{I}$: (144)

of $D^{II}+3D^{II}+6D^{II}+$ &c. to $n-2$ terms is $n.\frac{n-1}{2}.\frac{n-2}{3}D^{II}$:

of $D^{III}+4D^{III}+10D^{III}+$ &c. to $n-3$ terms is $n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}D^{III}$:

&c. &c.

And the aggregate must be the sum of n terms of the series $a+b+c+$ &c.

viz. $na+n.\frac{n-1}{2}D^{I}+n.\frac{n-1}{2}.\frac{n-2}{3}D^{II}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}D^{III}+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}.\frac{n-4}{5}D^{IV}+$ &c.

When the differences are finally $=o$, any term, or the sum of any number of the terms may be accurately determined; but if the differences do not vanish, the result is only an approximation: this approximate value however, will become nearer and nearer the truth as the differences diminish.

Examples.

1. What is the 17*th.* term of the series 1, 3, 6, 10, 15, &c.?

	1	3	6	10	15
1*st.* difference.....	2	3	4	5	
2d....................		1	1	1	
			0	0	

Here $a=1$, $D^{I}=2$, $D^{II}=1$; these being substituted in the expression $a+(n-1)D^{I}+\frac{n-1}{1}.\frac{n-2}{2}D^{II}$ &c. give $1+(n-1)\times 2+\frac{n-1}{1}\frac{n-2}{2}\times 1=\frac{n^2\times 4}{2}=$ (when $n=17$) $\frac{17^2+4}{2}=153$, the term required.

2. To find the *n*th. term of the series of rectangles 1×2, 3×4, 5×6, 7×8, 9×10, &c.

$$\begin{array}{lccccccc}
 & 2 & & 12 & & 30 & & 56 \\
\text{1st. diff.} \ldots\ldots\ldots\ldots & & 10 & & 18 & & 26 & \\
\text{2d.} \ldots\ldots\ldots\ldots\ldots\ldots & & & 8 & & 8 & & \\
 & & & & 0 & & &
\end{array}$$

Here $a=2$, $D^{\text{I}}=10$, $D^{\text{II}}=8$

And $2+(n-1)\times 10+\frac{n-1}{1}.\frac{n-2}{2}\times 8=4n^2-2n$ the required term.

3. To find the sum of n terms of the series of cubes $1^3+2^3+3^3+4^3+$ &c.

$$\begin{array}{lccccccccc}
 & 1 & & 8 & & 27 & & 64 & & 125 \\
\text{1. diff.} \ldots\ldots\ldots\ldots & & 7 & & 19 & & 37 & & 61 & \\
\text{2. diff.} \ldots\ldots\ldots\ldots & & & 12 & & 18 & & 24 & & \\
\text{3, diff.} \ldots\ldots\ldots\ldots & & & & 6 & & 6 & & & \\
 & & & & & 0 & & & &
\end{array}$$

In this example $a=1$, $D^{\text{I}}=7$, $D^{\text{II}}=12$, $D^{\text{III}}=6$;

And $n+n.\frac{n-1}{2}\times 7+n.\frac{n-1}{2}.\frac{n-2}{3}\times 12+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}\times 6=\frac{n^4+2n^3+n^2}{4}=\left(\frac{n^2+n}{2}\right)^2$ the expression for the sum required.

Hence it appears that the aggregate of any number of the series of *cubes* $1^3+2^3+3^3+4^3$ &c. taken in succession from 1, is a *square* number.

4. When the series is descending, the differences will be alternately *minus* and *plus*. Thus, to find the sum of the *biquadrates* $10^4+9^4+8^4+$ &c. to 8 terms:

$$\begin{array}{lccccccccccc}
 & 10000 & & 6561 & & 4096 & & 2401 & & 1296 & & 625 \\
\text{1. diff.} \ldots & & 3439 & & 2465 & & 1695 & & 1105 & & 671 & \\
\text{2. diff.} \ldots\ldots & & & 974 & & 770 & & 590 & & 434 & & \\
\text{3. diff.} \ldots\ldots\ldots & & & & 204 & & 180 & & 156 & & & \\
\text{4. diff.} \ldots\ldots\ldots\ldots & & & & & 24 & & 24 & & & & \\
 & & & & & & 0 & & & & &
\end{array}$$

Here $a=10000$, $D^{\mathrm{I}}=-3439$, $D^{\mathrm{II}}=+974$, $D^{\mathrm{III}}=-204$, $D^{\mathrm{IV}}=+24$, $n=8$,

And $10000\times 8-28\times 3439+56\times 974-70\times 204+56\times 24=25316$ the *sum required.*

And in the same manner, the sums of series of higher powers may be determined.

179. Hence we find

$$1^1+2^1+3^1\ldots\ldots\, n^1=\frac{n^2+n}{2}.$$

$$1^2+2^2+3^2\ldots\ldots\, n^2=\frac{2n^3+3n^2+n}{6}.$$

$$1^3+2^3+3^3\ldots\ldots\, n^3=\frac{n^4+2n^3+n^2}{4}.$$

$$1^4+2^4+3^4\ldots\ldots\, n^4=\frac{6n^5+15n^4+10n^3-n}{30}.$$

$$1^5+2^5+3^5\ldots\ldots\, n^5=\frac{2n^6+6n^5+5n^4-n^2}{12}.$$

&c. &c.

If we suppose n to be infinite, all its inferior powers may be rejected as inconsiderable in respect of the greatest or highest power, because any power of an infinite quantity is the next inferior power taken an infinite number of times, and we shall get an expression for the sum of an infinite series of powers whose roots are in arithmetical progression, having an infinite or indefinitely small quantity for the common difference:

Thus, rejecting $3n^2+n$ in the expression for the sum of a series of squares, gives $\frac{2n^3}{6}$ or $\frac{n^3}{3}=$ the sum of an infinite series of squares proceeding from o^2 the least, to n^2 the greatest.

And the sum of such a series of cubes will be $\frac{n^4}{4}$, the greatest term or cube being n^3:

of a series of biquadrates, $\frac{6n^5}{30}$ or $\frac{n^5}{5}$, the greatest being n^4:

of a series of 5*th.* powers, $\frac{2n^6}{12}$ or $\frac{n^6}{6}$, where the greatest is n^5.

&c. &c.

Hence it appears that the expression for the sum is found by adding 1 to the index, and dividing the power so increased by its index:

$$\begin{array}{rll} \text{Thus, the sum of the series of squares is} & \dfrac{n^3}{3} \text{ or } \dfrac{n^{2+1}}{2+1}: \\ \text{of the series of cubes....} & \dfrac{n^4}{4} \text{ or } \dfrac{n^{3+1}}{3+1}: \\ \text{of the biquadrates......} & \dfrac{n^5}{5} \text{ or } \dfrac{n^{4+1}}{4+1}. \\ \text{\&c.} & \text{\&c.} \end{array}$$

We therefore conclude that the sum of an infinite series of n^r powers will be $\frac{n^{r+1}}{r+1}$; where n^r is the greatest, and o^r the least terms of the series.

The differential method is also applied to the interpolation of series, the quadrature of curves, &c.

ON THE REVERSION OF SERIES.

180. WHEN the value of the root or unknown quantity in the terms of an infinite series is expressed by another infinite series in which that root is not found, the series is said to be reversed.

Thus, suppose $ay + by^2 + cy^3 + dy^4 + \text{\&c.} = x$; and let it be required to revert the series, or, to find y in an infinite series expressed in powers of x with coefficients.

By transposition, $ay + by^2 + cy^3 + dy^4 + \text{\&c.} - x = o$.

Assume $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c.

$$\begin{array}{l} \text{Then } y^2 = (Ax + Bx^2 + Cx^3 \text{ \&c.})^2 = A^2x^2 + 2ABx^3 + 2ACx^4 + B^2x^4 \text{ \&c.} \\ \quad y^3 = (Ax + Bx^2 + Cx^3 \text{ \&c.})^3 = A^3x^3 + 3A^2Bx^4 \text{ \&c.} \\ \quad y^4 = (Ax + Bx^2 + Cx^3 \text{ \&c.})^4 = A^4x^4 \text{ \&c.} \end{array}$$

$$\begin{array}{l} \text{And } ay = aAx + aBx^2 + aCx^3 + aDx^4 \text{ \&c.} \\ \quad +by^2 = \ldots\ldots + bA^2x^2 + 2bABx^3 + \left.\begin{array}{l} 2bACx^4 \\ +bB^2x^4 \text{ \&c.} \end{array}\right\} \\ \quad +cy^3 = \ldots\ldots\ldots\ldots\ldots\ldots + cA^3x^3 \quad + 3cA^2Bx^4 \text{ \&c.} \\ \quad +dy^4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + dA^4x^4 \text{ \&c.} \\ \quad -x = -x \\ \hline \left.\begin{array}{l} (aA - 1)x + (aB + bA^2)x^2 + (aC + 2bAB + cA^3)x^3 \\ \qquad + (aD + 2bAC + bB^2 + 3cA^2B + dA^4)x^4 \text{ \&c.} \end{array}\right\} \text{sum} \end{array}$$

Now this sum is $=ay+by^2+cy^3+dy^4$ &c. $-x=o$ (or equal to o); but to make the whole expression $=o$, the coefficients of x and its powers must vanish, or each become $=o$,

that is $aA-1=o$; whence $A=\frac{1}{a}$.

$aB+bA^2=o$; whence $B=-\frac{bA^2}{a}=-\frac{b}{a^3}$.

$aC+2bAB+cA^3=o$; whence $C=-\frac{2bAB}{a}-\frac{cA^3}{a}=\frac{2b^2-ac}{a^5}$.

$aD+2bAC+bB^2+3cA^2B+dA^4=o$; and $D=\frac{5abc-5b^3-a^2d}{a^7}$.

Therefore $y=\frac{1}{a}x-\frac{b}{a^3}x^2+\frac{2b^2-ac}{a^5}x^3-\frac{5b^3-5abc+a^2d}{a^7}x^4$ &c.

To revert the series $y+\frac{1}{2}y^2+\frac{1}{3}y^3+\frac{1}{4}y^4+\frac{1}{5}y^5$ &c. $=x$.

Here $a=1$, $b=\frac{1}{2}$, $c=\frac{1}{3}$, $d=\frac{1}{4}$, $f=\frac{1}{5}$, &c. these being substituted in the above series, we have

$y=x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4$ &c.

The law of this series for the value of y is not, perhaps, sufficiently evident from the coefficients 1, $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{24}$; but by extending the powers of ($Ax+Bx^2+Cx^3+Dx^4$ &c.) to another term, we get $E=+\frac{1}{120}$ the coefficient of the 5*th.* term;

hence $y=x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{120}x^5$ &c.

It therefore appears that the coefficient of the nth. term is the product $\frac{1}{1}\times\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}$ &c. continued to n factors, in this *case.*

If the series to be reversed is $y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\frac{1}{5}y5$ &c. $=x$;

then $a=1$, $b=-\frac{1}{2}$, $c=\frac{1}{3}$, $d=-\frac{1}{4}$, $f=\frac{1}{5}$,

and $y=x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5$, &c.

181. Suppose it is required to revert the series $ay+by^3+cy^5+dy^7$, &c.

Let $ay + by^3 + cy^5 + dy^7$, &c. $= x$: then $ay + by^3 + cy^5 + dy^7$ &c. $- x = o$.

Assume $y = Ax + Bx^3 + Cx^5 + Dx^7$ &c.

$$\begin{array}{lllll}
\text{Then } ay = aAx & + aBx^3 & + aCx^5 & + aDx^7 \text{ \&c.} & \\
by^3 = \ldots\ldots & + bA^3x^3 & + 3bA^2Bx^5 & \left.\begin{array}{l} + 3bA^2Cx^7 \\ + 3bAB^2x^7 \text{ \&c.} \end{array}\right\} & \\
cy^5 = \ldots\ldots\ldots\ldots\ldots & & + cA^5x^5 & + 5cA^4Bx^7 \text{ \&c.} & \\
dy^7 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & + dA^7x^7 \text{ \&c.} & \\
-x = -x & & & &
\end{array}$$

$$\text{sum} \left\{\begin{array}{l} (aA - 1)\,x + (aB + bA^3)\,x^3 + (aC + 3bA^2B + cA^5)\,x^5 \\ \qquad + (aD + 3bA^2C + 3bAB^2 + 5cA^4B + dA^7)\,x^7 \text{ \&c.} \end{array}\right.$$

Now making the coefficients each $= o$, as in the preceding article, we have

$aA - 1 = o$, whence $A = \frac{1}{a}$:

$aB + bA^3 = o$, whence $B = -\frac{b}{a^4}$:

$aC + 3bA^2B + cA^5 = o$ ..... and $C = \frac{3b^2 - ac}{a^7}$:

$aD + 3bA^2C + 3bAB^2 + 5cA^4B + dA^7 = o$, whence $D = -\frac{a^2d - 8abc + 12b^3}{a^{10}}$;

consequently $y = \frac{1}{a}x - \frac{b}{a^4}x^3 + \frac{3b^2 - ac}{a^7}x^5 - \frac{a^2d - 8abc + 12b^3}{a^{10}}x^7$ &c.

Examp. To revert the series $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7$ &c. $= x$.

Here $a = 1$, $b = \frac{1}{6}$, $c = \frac{3}{40}$, $d = \frac{5}{112}$, these being substituted, give $y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7$ &c. where the law of continuation is manifest; the coefficient of any term being the product $\frac{1}{1} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{4}$ &c. extended to as many factors as there are units in the index of x in that term: thus the continued product of the 5 factors $\frac{1}{1} \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{5}$ is $\frac{1}{120}$ the coefficient of x^5: And the coefficient of the 5*th.* term (or of x^9) will be the continued product of the 9 factors $\frac{1}{1} \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{5} \times \frac{1}{6} \times \frac{1}{7} \times \frac{1}{8} \times \frac{1}{9} = \frac{1}{362880}$, &c.

What we have given in the last ten articles respecting Series, may serve as an introduction to the study of those particular branches of the subject, which is one of the most copious and intricate in the science of Algebra.

OF CUBIC EQUATIONS.

182. AN Equation is said to have as many roots as there are units in the highest dimension of the unknown quantity:

Thus, if $x^2 = a^2$, then (129) x will have two values, for it may be either $+a$, or $-a$.

Let $x^3 - (a+b+c)x^2 + (ab+ac+bc)x = abc$,

or $x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = o$ be a cubic equation generated by the continued product $(x-a) \times (x-b) \times (x-c)$:

Then it is evident x may be taken equal to a, b, or c, for if either of those values be substituted for x, the whole expression will vanish; and consequently the equation must have three roots, or, there will be three values of x.

To illustrate this in numbers, let $a=2$, $b=3$, $c=5$; then the equation becomes $x^3 - 10x^2 + 31x = 30$,

or $x^3 - 10x^2 + 31x - 30 = 0$:

And if $x=2$, then $2^3 - 10 \times 2^2 + 31 \times 2 = 30$

if $x=3$........ $3^3 - 10 \times 3^2 + 31 \times 3 = 30$

if $x=5$........ $5^3 - 10 \times 5^2 + 31 \times 5 = 30$

Therefore 2, 3, and 5 are the three roots or values of x.

In the preceding equations the coefficient of the second term is the sum of the three roots, $2+3+5$:

The coefficient of the third term, the sum of their products taken two by two, $2 \times 3 + 2 \times 5 + 3 \times 5$:

And the last term their continued product, $2 \times 3 \times 5$.

183. If $(x+a)\times(x+b)\times(x+c)=o$, or $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc=o$ be the equation,

then the three roots will evidently be negative,

that is, $x=-a$, $x=-b$, $x=-c$.

Hence also,

if $(x+a)\times(x-b)\times(x-c)=o$, the three roots are $-a, +b, +c$:
and $(x+a)\times(x+b)\times(x-c)=o$, its roots are $-a, -b, +c$.
&c. &c.

It therefore appears that *cubic equations* may have all the roots positive, or all negative, or two may be negative and one positive, or two may be positive and one negative.

184. When one of the roots is discovered, the others may be found by depressing the equation, thus,

Let $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=o$,
or $(x-a)\times(x-b)\times(x-c)=o$, be the equation, (182),
and suppose $+a$ is found to be one of the roots or values of x; subtract this from x and we have $x-a$, then if the whole equation, or $(x-a)\times(x-b)\times(x-c)=o$ be divided by $x-a$, the quotient is $(x-b)\times(x-c)=o$, or $x^2-bx-cx+bc=o$, a quadratic equation which will give the other two roots.

If the root first discovered is negative (suppose $-a$ for example), then subtracting $-a$ from x, the divisor becomes $x+a$ instead of $x-a$.

To exemplify this in numbers: Let the equation be $x^3-10x^2+31x=30$, or $x^3-10x^2+31x-30=o$; and suppose $+2$ is found to be one of the roots; subtract this from x, and we have $x-2$ the divisor:

$$x-2)\ x^3-10x^2+31x-30\ (x^2-8x+15\ \text{a } \textit{quadratic.}$$

$$\begin{array}{l}
\underline{x^3-2x^2} \\
-8x^2+31x \\
\underline{-8x^2+16x} \\
\quad +15x-30 \\
\quad \underline{+15x-30} \\
\qquad 0
\end{array}$$

therefore $x^2-8x+15=o$, whence $x^2-8x=-15$, and $x=4\pm1=5$ and 3 the other two roots.

Again, if $x^3+2x^2-23x=60$, or $x^3+2x^2-23x-60=o$ be the equation, and -3 one of the roots; then $x+3$ is the divisor:

$$\begin{array}{rl}
x+3) & x^3+2x^2-23x-60\ (x^2-x-20, \textit{ a quadratic.} \\
 & \underline{x^3+3x^2} \\
 & \quad -x^2-23x \\
 & \quad \underline{-x^2-\ \ 3x} \\
 & \qquad\quad -20x-60 \\
 & \qquad\quad \underline{-20x-60} \\
 & \qquad\qquad\quad 0
\end{array}$$

And $x^2-x-20=0$, or $x^2-x=20$, whence $x=\frac{1}{2}\pm4\frac{1}{2}=+5$ and -4 the other two roots.

185. *To take away the second term from an Equation.*

SUPPOSE $x^2+2ax=b$, where x is the unknown quantity; and let $z-a=x$:

Then $(z-a)^2+2a(z-a)=z^2-2az+a^2+2az-2a^2=z^2-a^2=b$, or $z^2=b+a^2$, a simple quadratic in which z is the unknown quantity. Now $z=\surd(b+a^2)$, and $z-a=\surd(b+a^2)-a=x$, the same value of x as is found by completing the square in the given equation $x^2+2ax=b$.

Again, if $x^3+3ax^2=b$; then putting $z-a=x$, we have $(z-a)^3+3a(z-a)^2=b$;

$$\begin{array}{rl}
(z-a)^3= & z^3-3az^2+3a^2z-a^3 \\
3a(z-a)^2= & \underline{\qquad\ +3az^2-6a^2z+3a^3} \\
\text{sum} & \underline{z^3 \qquad\qquad -3a^2z+2a^3=b}
\end{array}$$

an equation in which z^2 the second power of the unknown quantity z is wanting.

And if the equation was $x^4+4ax^3=b$, the assumed value of x will be $z-a$: *We therefore divide the coefficient of the second term of the equation by the index of the highest power of the unknown quantity, and the quotient is the second member of the assumed root;* but when the second term in the equation is negative, that quotient must be positive.

$$\text{Thus, if } \left.\begin{array}{r} x^2 - 2ax = b \\ x^3 - 3ax^2 = b \\ x^4 - 4ax^3 = b \end{array}\right\} \text{ then } z + a = x.$$

Let the proposed equation be $x^3 - 12x^2 + 3x = -72$:

Then 12 (the coefficient of the second term) divided by 3 (the index of the highest power of x) gives 4; therefore assume $z + 4 = x$:

$$\begin{array}{rl} \text{then } (z+4)^3 = & z^3 + 12z^2 + 48z + 64 \\ -12\,(z+4)^2 = & \quad\;\; - 12z^2 - 96z - 192 \\ 3\,(z+4) = & \qquad\qquad\quad + 3z + 12 \\ \hline & z^3 \qquad\quad - 45z - 116 = -72, \\ \hline \end{array}$$

an equation where z^2, or the *2d.* term is wanting.

186. In a simple cubic equation when one root is rational, the other two are imaginary or impossible:

Thus, if $x^3 = 1$, or $x^3 - 1 = 0$; then $x = 1$ the rational root:

To find the other roots, we have $x - 1$ for the divisor (184);

whence $\dfrac{x^3 - 1}{x - 1} = x^2 + x + 1$ a *quadratic equation:*

And $x^2 + x + 1 = o$, or $x^2 + x = -1$, and completing the square, we have $x^2 + x + \frac{1}{4} = -1 + \frac{1}{4} = -\frac{3}{4}$, whence $x + \frac{1}{2} = \sqrt{(-\frac{3}{4})}$, and $x = \sqrt{(-\frac{3}{4})} - \frac{1}{2}$ one of the impossible roots:

But $-x - \frac{1}{2}$ is also the square root of $x^2 + x + \frac{1}{4}$, whence $-x - \frac{1}{2} = \sqrt{(-\frac{3}{4})}$, or $x = -\sqrt{(-\frac{3}{4})} - \frac{1}{2}$ the other:

Therefore the three values of x, or the three cube roots of 1, are 1, $\sqrt{(-\frac{3}{4})} - \frac{1}{2}$, and $-\sqrt{(-\frac{3}{4})} - \frac{1}{2}$.

Here follows the operation of cubing the last of the preceding roots;

$$\begin{array}{l} -\sqrt{(-\frac{3}{4})} - \frac{1}{2} \\ -\sqrt{(-\frac{3}{4})} - \frac{1}{2} \\ \hline \qquad -\frac{3}{4} + \frac{1}{2}\sqrt{(-\frac{3}{4})} \\ \qquad\quad \frac{1}{2}\sqrt{(-\frac{3}{4})} + \frac{1}{4} \\ \hline \qquad -\frac{3}{4} + \sqrt{(-\frac{3}{4})} + \frac{1}{4} \text{ the } \textit{square.} \\ \qquad -\sqrt{(-\frac{3}{4})} - \frac{1}{2} \\ \hline \qquad \frac{3}{4}\sqrt{(-\frac{3}{4})} - (-\frac{3}{4}) - \frac{1}{4}\sqrt{(-\frac{3}{4})} \\ \qquad \frac{3}{8} - \frac{1}{2}\sqrt{(-\frac{3}{4})} - \frac{1}{8} \\ \hline \qquad + \frac{3}{8} + \frac{3}{4} - \frac{1}{8} = 1 \text{ the } \textit{cube.} \\ \hline \end{array}$$

And in the same manner it may be shown that the cube of the other imaginary root is also = 1.

*CARDAN's *method of solving Cubic Equations.*

186. If the equation contains all the terms with coefficients, the whole equation must be divided by the coefficient of the highest power of the unknown quantity, and the second term taken away (185), it will then be reduced to this form $x^3 \pm ax = \pm b$.

Let the equation be $x^3+ax=b$; and put $y+z=x$:

$$\text{Then } (y+z)^3+a(y+z)=y^3+3y^2z+3yz^2+z^3+ay+az$$
$$=y^3+z^3+3yz(y+z)+a(y+z)=b:$$

Let $3yz=-a$ which substituted for $3yz$

then $y^3+z^3+3yz(y+z)+a(y+z)=b$

becomes $y^3+z^3-a(y+z)+a(y+z)=y^3+z^3=b$:

And since $3yz=-a$, we have $z=\frac{-a}{3y}$, and $z^3=\frac{-a^3}{27y^3}$ which put for z^3 in the equation $y^3+z^3=b$ gives $y^3+\frac{-a^3}{27y^3}=b$, whence $y^6-by^3=\frac{1}{27}a^3$, an equation of the quadratic form: and by completing the square we get $y^3=\frac{1}{2}b\pm\sqrt{\left(\frac{1}{27}a^3+\frac{1}{4}b^2\right)}$, and $y=\left(\frac{1}{2}b\pm\sqrt{\left(\frac{1}{27}a^3+\frac{1}{4}b^2\right)}\right)^{\frac{1}{3}}$.

And since $y^3+z^3=b$, z^3 is $=b-y^3$,

that is, $z^3=b-\left(\frac{1}{2}b\pm\sqrt{\left(\frac{1}{27}a^3+\frac{1}{4}b^2\right)}\right)$

or $z^3=\frac{1}{2}b\mp\sqrt{\left(\frac{1}{27}a^3+\frac{1}{4}b^2\right)}$

whence $z=\left(\frac{1}{2}b\mp\sqrt{\left(\frac{1}{27}a^3+\frac{1}{4}b^2\right)}\right)^{\frac{1}{3}}$

therefore

$$x\,(=y+z)=\left(\tfrac{1}{2}b\pm\sqrt{\left(\tfrac{1}{27}a^3+\tfrac{1}{4}b^2\right)}\right)^{\frac{1}{3}}+\left(\tfrac{1}{2}b\mp\sqrt{\left(\tfrac{1}{27}a^3+\tfrac{1}{4}b^2\right)}\right)^{\frac{1}{3}}.$$

Remark. When a is negative and $\frac{1}{4}b^2$ less than $\frac{1}{27}a^3$, the solution by this rule cannot generally be obtained, because the

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This method of cubic equations attributed to *Cardan* was discovered, it seems, by *Scipio Ferreus*, and *Nic. Tartalea.* All three were Italian mathematicians who flourished rather early in the 16*th.* century.

quantity $\left(\frac{1}{27}a^3 + \frac{1}{4}b^2\right)$ becomes negative, and therefore its square root $\sqrt{\left(\frac{1}{27}a^3 + \frac{1}{4}b^2\right)}$ is impossible. This is called the *irreducible case.*

Examples.

1. To find x in the equation $3x^3 + 18x^2 + 6x = 261$.

Divide the whole equation by 3 the coefficient of x^3, and we have $x^3 + 6x^2 + 2x = 87$.

Next, to take away the 2*d.* term ($6x^2$), let $z - \frac{6}{3}$ or $z - 2 = x$ (185):

$$\begin{array}{rl} \text{then } (z-2)^3 = & z^3 - 6z^2 + 12z - 8 \\ 6(z-2)^2 = & + 6z^2 - 24z + 24 \\ 2(z-2) = & + 2z - 4 \\ \hline & z^3 - 10z + 12 = 87, \text{ or } z^3 - 10z = 75. \end{array}$$

Here $a = -10$, and $b = 75$, these values being substituted in the general expression give

$$\left(\frac{75}{2} \pm \sqrt{\left(\frac{-10^3}{27} + \frac{75^2}{4}\right)}\right)^{\frac{1}{3}} + \left(\frac{75}{2} \mp \sqrt{\left(\frac{-10^3}{27} + \frac{75^2}{4}\right)}\right)^{\frac{1}{3}} = z$$

$$\text{or } \left(37\tfrac{1}{2} \pm \sqrt{1369\tfrac{23}{108}}\right)^{\frac{1}{3}} + \left(37\tfrac{1}{2} \mp \sqrt{1369\tfrac{23}{108}}\right)^{\frac{1}{3}} = z$$

where both the upper or both the lower signs may be taken in all cases, therefore retaining the former, we have

$$\left(37\tfrac{1}{2} + \sqrt{1369\tfrac{23}{108}}\right)^{\frac{1}{3}} + \left(37\tfrac{1}{2} - \sqrt{1369\tfrac{23}{108}}\right)^{\frac{1}{3}} = z$$

$$\text{or } 4.207 + .793 = 5 = z:$$

Now $z - 2 = x$, whence $x = 3$ the value of x in the given equation.

2. Let the equation be $x^3 + 8x = -399$, to find x.

Here $a = +8$, and $b = -399$; whence, by substitution

$$x = \left(\frac{-399}{2} + \sqrt{\left(\frac{512}{27} + \frac{159201}{4}\right)}\right)^{\frac{1}{3}} + \left(\frac{-399}{2} - \sqrt{\left(\frac{512}{27} + \frac{159201}{4}\right)}\right)^{\frac{1}{3}}$$

$$= \left(-199\tfrac{1}{2} + \sqrt{39819\tfrac{23}{108}}\right)^{\frac{1}{3}} + \left(-199\tfrac{1}{2} - \sqrt{39819\tfrac{23}{108}}\right)^{\frac{1}{3}} = 0.36$$

$-7.36 = -7$ the root, or value of x.

The other two roots may be found as in *Art.* 184, thus, $x^3+8x+399=0$, and $\frac{x^3+8x+399}{x+7}=x^2-7x+57=0$, this quadratic gives $x=3\frac{1}{2}\pm\sqrt{(-44\frac{3}{4})}$; therefore both are imaginary or impossible.

187. Sometimes the method of solving an equation may be discovered by the addition, or subtraction, of a given quantity:

Thus, suppose $x^3+6x^2+12x=504$, where it appears that if 8 be added to each side of the equation, the sums will be complete cubes;

$$x^3+6x^2+12x+8=512$$
$$\text{whence}\quad x+2=8,\ \text{and}\ x=6.$$

Again, if $x^3-12x^2+48x=100$, then by subtracting 64 $\left(\text{the cube of } \frac{12}{3}\right)$ we have $x^3-12x^2+48x-64=36$, and taking the cube roots, $x-4=36^{\frac{1}{3}}$, or $x=4+36^{\frac{1}{3}}$.

THE RESOLUTION OF EQUATIONS BY APPROXIMATION.

188. THE preceding methods of solution are restricted to particular cases which seldom occur in practice. We shall therefore proceed to the resolution of equations by approximation: And for this purpose the rule of Double Position or Trial-and-error (Arith. 109.) seems the most general and expeditious of any.

Examples.

1. To find x in the cubic equation $x^3+x^2=500$.

From a trial or two it appears that the value of x is between 7 and 8; therefore let those two numbers be the first assumptions:

$$\begin{array}{rrlrrl}
\text{then } 7^3 = & 343 & & 8^3 = & 512 & \\
7^2 = & \underline{49} & & 8^2 = & \underline{64} & \\
 & 392 & & & 576 & \\
 & \underline{500} & & & \underline{500} & \\
\textit{error} & 108 & \text{too } \textit{little.} & \textit{error} & 76 & \text{too } \textit{great.} \\
 & & & & \underline{108} & \\
 & & & \text{sum of errors} & \underline{184} &
\end{array}$$

$$\begin{array}{rr}
108 \times 8 = & 864 \\
76 \times 7 = & \underline{532} \\
184) & 1396 \\
\end{array}$$

quotient nearly $\overline{7.6}$ the first approximation.

$$\begin{array}{rl}
7.6^3 = & 438.976 \\
7.6^2 = & \underline{57.76} \\
 & 496.736 \\
 & \underline{500} \\
\text{too } \textit{little} & 3.264 \\
\end{array}$$

it therefore appears that the value of x is greater than 7.6

Now let 7.61 and 7.62 be the two suppositions:

$$\begin{array}{rlcrl}
\text{then } 7.61^3 = & 440.711081 & & 7.62^3 = & 442.450728 \\
7.61^2 = & \underline{57.9121} & & 7.62^2 = & \underline{58.0644} \\
 & 498.623181 & & & 500.515128 \\
 & \underline{500} & & & \underline{500} \\
\textit{error} & \underline{1.376819} \text{ too } \textit{little.} & & \textit{error} & \underline{0.515128} \text{ too } \textit{great.} \\
\end{array}$$

$$\begin{array}{rr}
1.376819 \times 7.62 = & 10.49136078 \\
.515128 \times 7.61 = & 3.92012408 \\
\end{array}$$

$$\begin{array}{rl}
 & 1.376819 \\
 & \underline{0.515128} \\
\text{sum} & \underline{1.891947} \; \ldots\ldots\ldots\ldots \; 1.891947) \; \underline{14.41148486} \text{ sum} \\
\end{array}$$

quotient 7.617 second approximation, which is the value of x nearly.

2. Let the equation be $x^3 - 50x^2 + 3x = -4103$; to find x?

By a few trials x is found to be greater than 10 but less than 11; now assuming these numbers, then

$$\begin{array}{rrcrr}
10^3 = & 1000 & & 11^3 = & 1331 \\
3 \times 10 = & \underline{30} & & 3 \times 11 = & \underline{33} \\
 & +\ 1030 & & & +\ 1364 \\
50 \times 10^2 = & \underline{-\ 5000} & & 50 \times 11^2 = & \underline{-\ 6050} \\
 & -\ 3970 & & & -\ 4686 \\
 & \underline{-\ 4103} & & & \underline{-\ 4103} \\
\textit{error} & 133 \text{ too } \textit{little} & & \textit{error} & \underline{583} \text{ too } \textit{great.} \\
 & \underline{583} & & & \\
\text{sum} & \underline{716} & & & \\
\end{array}$$

$$\begin{array}{rr}
10 \times 583 = & 5830 \\
11 \times 133 = & \underline{1463} \\
716) & \underline{7293} \text{ sum} \\
\end{array}$$

quotient nearly $\underline{10.2}$ first approximation.

Now assume $x = 10.2$		again, suppose $x = 10.18$	
$10.2^3 =$	1061.208	$10.18^3 =$	1054.977832
$3 \times 10.2 =$	30.6	$3 \times 10.18 =$	30.54
	$+1091.808$		$+1085.517832$
$50 \times 10.2^2 =$	-5202	$50 \times 10.18^2 =$	-5181.62
	-4110.192		-4096.102168
	-4103		-4103
error	7.192 too *great*	*error*	6.897832 too *little*.
	6.897832		
sum	14.089832		

$$10.2 \times 6.897832 = 70.3578864$$
$$10.18 \times 7.192 \ldots = 73.21456$$
$$14.089832)\ 143.5724464$$

quotient nearly 10.19 second approximation, which is very nearly the value of x.

To find the other two roots we have $x - 10.19$ for the *divisor* (184); and $x^3 - 50x^2 + 3x + 4103 = 0$, the *dividend*;

then

$$x - 10.19)\ x^3 - 50x^2 + 3x + 4103\ (x^2 - 39.81x - 402.6639 \text{ a } \textit{quadratic.}$$
$$\underline{x^3 - 10.19x^2}$$
$$-39.81x^2 + 3x$$
$$\underline{-39.81x^2 + 405.6639x}$$
$$-402.6639x + 4103$$
$$\underline{-402.6639x + 4103.145141}$$

By this operation the learner will perceive that the true value of x is somewhat less than 10.19, the error in the whole equation being the decimal .145141.

Now $x^2 - 39.81x - 402.6639 = 0$, whence $x^2 - 39.81x = 402.6639$, and by completing the square, &c. $x = 19.905 \pm 28.26434 = 48.16934$, and -8.35934 the other two roots. Therefore the equation has three possible roots, two positive, and one negative.

And the sum of the three roots, $10.19 + 48.16934 - 8.35934 = 50$ the coefficient of the second term. (182).

3. To find x in the biquadratic equation $x^4 - 10x^3 + 100x^2 - 70x = 42676$?

The value of x appears to be between 15 and 16, therefore assuming those numbers, we shall have

$$\begin{array}{rr} 15^4 = & 50625 \\ 100 \times 15^2 = & 22500 \\ \hline & +73125 \\ & -34800 \\ \hline & 38325 \\ & 42676 \\ \hline \textit{error} & 4351 \text{ too } \textit{little.} \end{array} \qquad \begin{array}{rr} 10 \times 15^3 = & 33750 \\ 70 \times 15 \;= & 1050 \\ \hline & -34800 \end{array}$$

$$\begin{array}{rr} 16^4 = & 65536 \\ 100 \times 16^2 = & 25600 \\ \hline & +91136 \\ & -42080 \\ \hline & 49056 \\ & 42676 \\ \hline \textit{error} & 6380 \text{ too } \textit{great} \\ & 4351 \\ \hline \text{sum} & 10731 \end{array} \qquad \begin{array}{rr} 10 \times 16^3 = & 40960 \\ 70 \times 16 \;= & 1120 \\ \hline & -42080 \\ \\ \\ 16 \times 4351 = & 69616 \\ 15 \times 6380 = & 95700 \\ \hline \text{sum} & 165316 \end{array}$$

$\frac{165316}{10731} = 15.4$ the first approximation for the value of x; now let this be assumed:

$$\begin{array}{rr} 15.4^4 = & 56244.8656 \\ 100 \times 15.4 \;= & 23716 \\ \hline & +79960.8656 \\ & -37600.64 \\ \hline & 42360.2256 \\ & 42676 \\ \hline \textit{error} & 315.7744 \text{ too } \textit{little} \end{array} \qquad \begin{array}{rr} 10 \times 15.4^3 = & 36522.64 \\ 70 \times 15.4 \;= & 1078 \\ \hline & -37600.64 \end{array}$$

error 4351........ too *little*, by assuming $x = 15$ as before.

diff. of errors...... 4035.2256

$15. \times 315.7744 = 4736.616$

$15.4 \times 4351 = 67005.4$

diff. of products 62268.784 whence $\frac{62268.784}{4035.2256} = 15.43$ the second approximation, which is the value of x nearly.

To depress the equation in order to approximate the other roots, the divisor is $x - 15.43$;

$$x-15.43)x^4-10x^3+100x^2-70x-42676=0(x^3+5.43x^2+183.7849x+2765.8$$

$$\begin{array}{l}
x^4-15.43x^3 \\
\hline
\quad +5.43x^3+100x^2 \\
\quad +5.43x^3-83.7849x^2 \\
\hline
\qquad +183.7849x^2-70x \\
\qquad +183.7849x^2-2835.801007x \\
\hline
\qquad\qquad +2765.801007x-42676 \\
\qquad\qquad +2765.801007x-42676.3 \text{ \&c.} \\
\hline
\qquad\qquad\qquad 0
\end{array}$$

Now $x^3+5.43x^2+183.7849x+2765.8=0$, or $x^3+5.43x^2+183.7849x=-2765.8$ where x is evidently negative; and from a few trials its value appears to be a little greater than 11; if therefore -11.1 and -11.2 are made the first suppositions, two operations will bring out $x=-11.163$ one of the roots. Consequently $x+11.163$ is the divisor for depressing the cubic to a quadratic equation:

Hence $\dfrac{x^3+5.43x^2+183.7849x+2765.8}{x+11.163}=x^2-5.733x+247.782379=0$,

or $x^2-5.733x=-247.782379$, whence $x=2.8665\pm\sqrt{(-239.56 \text{ \&c.})}$ which values are both impossible or imaginary. The equation therefore, has a positive, a negative, and two impossible roots.

$$\begin{array}{l}
+15.43 \\
-11.163 \\
+\ 2.8665+\sqrt{(-239.56 \text{ \&c.})} \\
+\ 2.8665-\sqrt{(-239.56 \text{ \&c.})} \\
\hline
10.0000
\end{array}$$

sum of roots, equal to the coefficient of the second term in the given equation.

4. Let the proposed equation be $(7x^3+4x^2)^{\frac{1}{3}}+(20x^2-10x)^{\frac{1}{2}}=28$; to find the value of x?

By trial x is found to be between 4 and 5, therefore let those numbers be the first suppositions:

$(7\times4^3+4\times4^2)^{\frac{1}{3}}=8$	$(7\times5^3+4\times5^2)^{\frac{1}{3}}=9.916$
$(20\times4^2-10\times4)^{\frac{1}{2}}=16.73$	$(20\times5^2-10\times5)^{\frac{1}{2}}=21.213$
24.73	31.129
28	2
error too *little* 3.27	*error* too *great* 3.129
	3.27
	sum 6.399

$4 \times 3.129 = 12.516$

$5 \times 3.27 = 16.35$

sum of products.......... $\underline{28.866}$ $\frac{28.866}{6.399} = 4.5$ the first approximation.

Next, assuming $x = 4.5$ and 4.51, and repeating the operation, we get 4.511, 4.5107, 4.51066 the successive values of x, the last being a very near approximation.

N. B. In resolving these complex equations, the student should make use of logarithms for raising powers and extracting roots, otherwise he will find the operation extremely tedious.

189. *Logarithms* are also peculiarly adapted to the resolution of *exponential* equations. We shall subjoin a few examples.

1. Suppose $3^x = 19683$; to find x?

It is evident from Arith. *art.* 187, that the *log.* of 3 multiplied by the exponent x gives the *log.* of 19683;

that is $x \times$ *log.* of $3 =$ *log.* of 19683,

therefore $x = \frac{\textit{log.}\text{ of }19683}{\textit{log.}\text{ of }3}$, or $\frac{4.294091}{0.477121} = 9$ the value of x: for $3^9 = 19683$.

2. To find x in the equation $24^x = 44620$?

$x = \frac{\textit{log.}\ 44620}{\textit{log.}\ 24} = \frac{4.649530}{1.380211} = 3.368709$ &c. This value of x however, is too great: the error arises in consequence of using logarithms to 6 places of decimals only; for the result by logarithms to 10 places will be 3.368708662 &c. and since 3.368709 &c. is correct in the *6th.* figure, the other value from logarithms to 10 places will probably be so in the *10th.*

3. Let the equation be $x^x = 46060$, to find the value of x?

$6^6 = 46656$, hence x appears to be a little less than 6:

Therefore let 6 and 5.9 be the two first assumptions for x:

$6^6 = 46656$

$\underline{46060}$

error 596 too *great*

5.9 *log.* 0.770852

$5.9 \times 0.770852 = 4.5480268$ the *log.* of $35325 = 5.9^{5.9}$

$\underline{46060}$

error 10735 too *little.*

$\underline{596}$

sums of errors 11331

$6 \times 10735 = 64410$
$5.9 + 596 = 3516.4$
sum of products $= 67926.4$

And $\frac{67926.4}{11331} = 5.99$ the first approximate value of x.

Again, let 6, and 5.99 be the next suppositions:

5.99 *log.* 0.777427

and $5.99 \times 0.777427 = 4.656787$ &c. the *log.* of $45372 = 5.99^{5.99}$

46060
error 688 too *little*
supposition 6……*error* 596 too *great*
sum of errors 1284

$6 \times 688 = 4128$
$5.99 \times 596 = 3570.04$
sum 7698.04 and $\frac{7698.04}{1284} = 5.995$ the 2*d.* approximation.

Next, suppose 6, and 5.995 are the assumptions:

5.995 *log.* 0.777789

and $5.995 \times 0.777789 = 4.662845$ &c. the *log.* of $46009 = 5.995^{5.995}$

46060
error 51 too *little*
supposition 6, *error* 596 too *great*
sum 647

$6 \times 51 = 306$
$5.995 \times 596 = 3573.02$
sum products 3879.02 $\frac{3879.02}{647} = 5.9954$ the 3*d.* approximation.

Again, suppose 5.995 and 5.9954

5.9954 *log.* 0.777818

$5.9954 \times 0.777818 = 4.663330$ &c. the *log.* of $46060.6 = 5.9954^{5.9954}$

46060
error…… $.6$ too *great*
supposition 5.995, *error……* 51.0 too *little*
sum 51.6

$51 \times 5.9954 = 305.7654$
$.6 \times 5.995 = 3.597$
sum 309.3624 $\frac{309.3624}{51.6} = 5.995395$ the 4*th.* approximation, which is very nearly the true value of x: for a table of logarithms to 10 places gives $5.995395^{5.995395} = 46060.1$.

190. But the operation in this method of approximation may be somewhat abridged in the following manner:

If S and s be the two suppositions, D and d the corresponding errors, and x the number sought:

Then (Art. 128, *examp.* 8.)

$x - S : x - s :: D : d$, and by division (90)

$s - S : x - s :: D - d : d$

or $D - d : d :: s - S : x - s$ when the errors are alike.

And

$d : D : s - x : x - S$, and by composition (89)

$d + D : d :: s - S : s - x$, when the errors are unlike.

To apply this in the last example, we have

$$\left.\begin{array}{llll} s = 6 \ldots\ldots\ldots & d = & 596 & \text{error too } \textit{great} \\ S = 5.9 \ldots\ldots & D = & 10735 & \text{error too } \textit{little} \end{array}\right\} \text{unlike}$$

$$s - S = \overline{0.1} \qquad \text{sum } \overline{11331} = d + D$$

As 11331 : 596 :: 0.1 : 0.005, and $6 - 0.005 = 5.995$ first approximation.

Now let 6 and 5.995 be the two suppositions;

$$\begin{array}{llll} & 6 \ldots\ldots\ldots\ldots\ldots & \textit{error} \ 596 & \text{too } \textit{great} \\ & 5.995 \ldots\ldots\ldots & \textit{error} \ \underline{\ 51} & \text{too } \textit{little.} \\ \text{diff.} & \overline{0.005} & \text{sum} \ 647 & \end{array}$$

As $647 : 51 :: .005 : \dfrac{51 \times .005}{647} = .00039$, and $5.995 + .00039 = 5.99539$ second approximation: &c.

Remark. It is to be observed that the correction or 4*th.* term of the proportion, must always be applied to that assumed number which gives the error that is made use of in finding the correction: thus in the first proportion, .005 is the correction, and 596 the error that is used, and therefore the correction must be applied to the supposition 6; now 6 is too great, consequently the correction is subtractive: but if we take the other error, the correction must be added to the other supposition because that is defective,

Thus 11331 : 10735 :: 0.1 : .095 the correction, and $5.9 + .095 = 5.995$ the first approximation, as before.

The student will also perceive that when the errors are *alike*, their *difference* will be the first term of the proportion.

By raising 46060 to the 1000000th. power, the equation will be freed from the fractional index, thus, $9.995395 = \frac{9995395}{1000000}$, therefore $x^{\frac{9995395}{1000000}} = 46060$, and $x^{9995395} = 46060^{1000000}$, or $9.995395^{9995395} = 46060^{1000000}$.

191. *In particular cases, the unknown quantity in an equation may be found by summing a series.*

1. Thus, suppose $3x + 4x^2 + 7x^3 + 11x^4$ &c. *in infin.* $= 2$, where it is manifest the value of x must be less than 1; and by adding 1 to each side of the equation,

$1 + 3x + 4x^2 + 7x^3$ &c. $= 3$: now $1 + 3x + 4x^2 +$ &c. is a recurring series whose sum $= \frac{1 + 2x}{1 - x - x^2}$ (172) $= 3$, whence $1 + 2x = 3 - 3x - 3x^2$, which quadratic equation gives $x = \frac{1}{3}$.

2. Again, if $3x + 6x^2 + 10x^3 + 15x^4$, &c. *in infin.* $= 100$, then adding 1 to each side of the equation, we have $1 + 3x + 6x^2 + 10x^3$ &c. $= 101$; and the sum of the recurring series $1 + 3x + 6x^2$, &c. $= \frac{1}{1 - 3x + 3x^2 - x^3}$ or $\frac{1}{(1 - x)^{\frac{1}{3}}} = 101$, therefore $\frac{1}{1 - x} = 101^{\frac{1}{3}}$, whence $x = 1 - \frac{1}{101^{\frac{1}{3}}}$.

3. Suppose $x - x^2 + x^3 - x^4 + x^5$ &c. *in infin.* $= \frac{1}{3}$: then reversing the series, (180)

gives $x = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$ &c. which is a geometrical series, and the sum *ad infin.* $= \frac{1}{2}$, or the value of x.

The value of x however, may be found without reverting the series.

4. Let the equation be $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$ &c. *in infin.* $= 1$.

By reverting the series, $x = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} +$ &c.

5. Suppose the given equation is $\frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{1}{60}x^6 + \frac{1}{120}x^5 +$ &c. *in infin.* $= 8$; to find the value of x?

The coefficients constitute the series

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\frac{1}{5.6.7}\ \&c.$$

In order to investigate a theorem for the sums of such series, it may be observed, that in a series of quantities, the sum of all the differences of the terms taken in succession, is always equal to the difference between the first and last terms of the series:

Thus, if the series be $29+24+20+17+15$
differences $5+4+3+2=14=29-15$.

Or if $22+15+9+5+2+0$ be the series,
differences $7+6+4+3+2=22=22-0$: and so of others.

Now let $1+a+ab+abc+abcd+$&c. be a series of quantities which continually decrease so that the last term becomes indefinitely small or equal to o; then taking the differences of the terms, we have

$$\begin{aligned}
1-a &= 1-a\\
a-ab &= a\,(1-b)\\
ab-abc &= ab\,(1-c)\\
abc-abcd &= \underline{abc\,(1-d)}\ \&c.
\end{aligned}$$

sum of the differences $1-a+a\,(1-b)+ab\,(1-c)+abc\,(1-d)$ &c. $=1-0=1$ the difference of the first and last terms.

Now let a, b, c, &c. be expounded by fractional quantities,

viz. suppose $a=\frac{m}{v}$

$$b=\frac{m+p}{v+p} \qquad d=\frac{m+r}{v+r}$$

$$c=\frac{m+q}{v+q} \qquad f=\frac{m+s}{v+s}\ \&c.$$

then $1-a=1-\frac{m}{v}=\frac{v-m}{v+p}$

$$1-b=1-\frac{m+p}{v+p}=\frac{v-m}{v+p}$$

$$1-c=1-\frac{m+q}{v+q}=\frac{v-m}{v+q}$$

$$1-d=1-\frac{m+r}{v+r}=\frac{v-m}{v+r}\ \&c.$$

These several values being substituted in the equation

$$1 - a + a\,(1-b) + ab\,(1-c) + abc\,(1-d)\ \&\text{c. we have}$$

$$\frac{v-m}{v} + \frac{m}{v}\left(\frac{v-m}{v+p}\right) \frac{m}{v} \cdot \frac{m+p}{v+p}\left(\frac{v-m}{v+q}\right) + \frac{m}{v} \cdot \frac{m+p}{v+p} \cdot \frac{m+q}{v+q}\left(\frac{v-m}{v+r}\right)$$
$$\&\text{c.} = 1\,;$$

and dividing the whole equation by $\frac{v-m}{v}$ gives

$$1 + \frac{m}{v+p} + \frac{m}{v+p} \cdot \frac{m+p}{v+q} + \frac{m}{v+p} \cdot \frac{m+p}{v+q} \cdot \frac{m+q}{v+r}\ \&\text{c.} = \frac{v}{v-m}:$$

If $q = 2p$, $r = 3p$, $s = 4p$, &c. and $v + p = n$, then

$$1 + \frac{m}{n} + \frac{m}{n} \cdot \frac{m+p}{n+p} + \frac{m}{n} \cdot \frac{m+p}{n+p} \cdot \frac{m+2p}{n+2p} + \frac{m}{n} \cdot \frac{m+p}{n+p} \cdot \frac{m+2p}{n+2p} \cdot \frac{m+3p}{n+3p}$$
$$\&\text{c.} = \frac{n-p}{n-p-m}:\ \text{let } p = 1,\ \text{then}$$

$$1 + \frac{m}{n} + \frac{m\,(m+1)}{n\,(n+1)} + \frac{m\,(m+1)\,(m+2)}{n\,(n+1)\,(n+2)} + \frac{m\,(m+1)\,(m+2)\,(m+3)}{n\,(n+1)\,(n+2)\,(n+3)}$$
$$+\ \&\text{c. } in\ infin. = \frac{n-1}{+n-1-m},$$ which is a general theorem for summing a variety of series infinitely continued *.

To adapt this theorem to the series proposed, let $n = m + 3$, then dividing the whole equation by $m\,(m+1)\,(m+2)$, we get

$$\frac{1}{m\,(m+1)\,(m+2)} + \frac{1}{(m+1)\,(m+2)\,(m+3)} + \frac{1}{(m+2)\,(m+3)\,(m+4)} + \&\text{c.}$$
$$= \frac{1}{m\,(m+1)\,(2)},$$ which, when $m = 1$, becomes $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}$ $+ \frac{1}{4.5.6}$ &c. $= \frac{1}{4}$ the sum of the series infinitely continued.

Hence $\frac{1}{4}x^5 = 8$, and $x^5 = 32$, whence $x = 2$.

192. Though this subject of series is rather misplaced, we shall subjoin a few examples to show the use of the preceding theorem in the summation of particular series.

If $n = m + 4$, and the whole equation be divided by $m\,(m+1)\,(m+2)$ $(m+3)$, we have

$$\frac{1}{m\,(m+1)\,(m+2)\,(m+3)} + \frac{1}{(m+1)\,(m+2)\,(m+3)\,(m+4)}\ \&\text{c.} =$$
$$\frac{1}{m\,(m+1)\,(m+2)\,(3)},$$

---

* Simpson's Algebra.

where the law of continuation for the sums of these kind of series is evident: and hence it appears, that if the difference of the first and last factors in the denominator of the first term be substituted for the last factor, the resulting fraction will be the sum of the whole infinite series:

Thus, in the expression for the sum of $\frac{1}{6}+\frac{1}{24}+\frac{1}{60}$ &c. the first term is $\frac{1}{m\,(m+1)\,(m+2)}$, and the difference of m and $m+2$, the first and last factors in the denominator, is 2, which substituted for $m+2$ the last factor, gives $\frac{1}{m\,(m+1)\,(2)}$ the sum of the series.

This will enable us very readily to find the sum of any given number of terms of the series: for example, to determine the sum of the 20 first terms of the series $\frac{1}{6}+\frac{1}{24}+\frac{1}{60}$ &c.

The 21*st.* term is $\frac{1}{21\,.\,22\,.\,23}$, therefore substituting $23-21$ for 23 gives $\frac{1}{21\,.\,22\,.\,2}$ or $\frac{1}{924}$ the sum of the series infinitely continued when $\frac{1}{21\,.\,22\,.\,23}$ is the first term; now $\frac{1}{4}$ being the *whole* sum when $\frac{1}{1\,.\,2\,.\,3}$ is the first term, we have $\frac{1}{4}-\frac{1}{924}=\frac{115}{462}$ the 20 first terms.

In the general theorem if $m=1$, then

$$1+\frac{1}{n}+\frac{1\,.\,2}{n(n+1)}+\frac{1\,.\,2\,.\,3}{n\,(n+1)\,(n+2)}+\frac{1\,.\,2\,.\,3\,.\,4}{n\,(n+1)\,(n+2)\,(n+3)}\ \&c.=\frac{n-1}{n-2},$$

an expression for the several orders of the reciprocals of figurate numbers infinitely continued.

Thus, if $n=2$,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\ \&c.=\frac{3}{0},\ \text{therefore not summable.}$$

If $n=3$,

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}\ \&c.=2.$$

If $n=4$,

$$1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}\ \&c.=\frac{3}{2}.$$

And so on for other values of n.

OF SIMPLE INTEREST.

193. ALL the computations which relate to Simple Interest may be wrought arithmetically (Arith. *art.* 106); but algebraic theorems for the different cases will facilitate the practice considerably.

Let r = the *rate* of interest, or the interest of one pound for one year.
p = any principal or sum bearing interest.
t = the time in years.
a = the amount in the time t, or sum of the principal and interest.

Then $1 : r :: p : rp$ = the interest of the sum p for 1 year, and trp = the interest for the time t; whence $p + trp$ (the sum of the principal and interest) $= a$ the amount in the time t: this equation gives the following theorems:

$$r = \frac{a - p}{pt} \text{ the rate.}$$

$$p = \frac{a}{1 + rt} \text{ the principal.}$$

$$t = \frac{a - p}{pr} \text{ the time.}$$

$$a = p + trp = (1 + tr)p \text{ the amount.}$$

A few examples will show the use of these expressions.

1. What is the amount of 400*l.* in 5 months at 4 *per cent.*?

To find r: as 100*l.* : 4*l.* :: 1*l.* : .04 = r.

therefore $r = .04$
$p = 400$
$t = \frac{5}{12}$

And $a = p + trp = 400 + \frac{5}{12} \times .04 \times 400 =$ 406*l.* 13*s.* 4*d.* the sum required.

2. What is the interest of 1 pound for 1 day at 5 *per cent.*?

$$100 : 5 :: 1 : .05 = r:$$

Here then, we have $r = .05$

$p = 1$

$t = \frac{1}{365}:$

And a (the amount) $= (1 + tr)\,p = 1 + \frac{.05}{365} = 1.000136986$, &c. the amount, from which deducting the principal 1, there remains .000136986 &c. of a pound, the *answer.*

3. What sum in ready money is equivalent to 600*l.* due nine months hence, allowing 5 *per cent.* discount?

In this case $r = .05$

$t = \frac{9}{12}$

$a = 600$. And $p = \dfrac{600}{1 + .05 \times \frac{9}{12}} = \overset{l.}{578.313}$ the *answer.*

4. At what rate will 400*l.* in 18 months amount to, or raise a stock of 440*l.*?

Here $t = 1\frac{1}{2}$

$p = 400$

$a = 440$. And $r = \dfrac{440 - 400}{400 \times 1\frac{1}{2}} = \frac{40}{600} = \frac{1}{15}$ the *rate* or the interest of 1 pound for 1 year; whence $\frac{1}{15} \times 100 = 6\frac{1}{2}$*l.* *per cent.* the *answer.*

5. In what time will 360*l.* raise a stock of 370*l.* at 4 *per cent.*?

Here we have given $r = .04$

$p = 360$

$a = 370$. And $t = \dfrac{370 - 360}{360 \times .04} = 253\frac{1}{2}$ *days*, nearly the time required.

OF COMPOUND INTEREST.

194. THEOREMS for the solution of the different cases of compound interest may be derived from a process similar to that for finding the amount of a given sum in a given time, Arith. *art.* 107.

In these computations the amount of 1 pound in 1 year at *simple interest* is usually called the *rate:*

Thus, the amount of 100*l.* in 1 year at 5 *per cent.* is 105*l*, then 100 : 105 : : 1 : 1.05 (the amount of 1 pound in 1 year) is the *rate:* and 1.04 is the *rate* when 4 *per cent.* is the interest.

Let $r =$ the rate,
$p =$ any principal,
$t =$ the time in years,
$a =$ the amount.

To find the amount of any principal sum (p) for the time t.

As $\overset{l.}{1} : r :: \overset{l.}{p} : rp$ the amount of p pounds at the end of 1 year.

And $1 : r :: rp : r^2p$ the amount of rp pounds at the end of the 2*d.* year.

Also $1 : r :: r^2p : r^3p$ the amount in three years, &c.

Hence it appears that $r^t p$ will be the amount in t years,

viz, $r^t p = a$. Whence the following theorems are readily obtained:

$$a = r^t p. \qquad r = \left(\frac{a}{p}\right)^{\frac{1}{t}}.$$

$$p = \frac{a}{r^t}. \qquad t = \frac{\log. \frac{a}{p}}{\log. r}. \text{ (189. } ex. \text{ 1.)}$$

By means of logarithms these expressions are simple in their application.

Examples.

1. What is the compound interest of 200*l.* in 15 years at 5 *per cent.*?

Here $r = 1.05$, $p = 200$, $t = 15$:

$$\begin{array}{rrl}
r = 1.05\ldots\ldots\ldots \textit{log.} & 0.021189 & \\
t = 15\ldots\ldots\ldots\ldots & 15 & \\
\hline
 & 105945 & \\
 & 21189 & \\
\hline
 & 0.317835 & \textit{log.}\, r^t \\
p = 200\ \textit{log.} & 2.301030 & \\
\hline
\text{the amount}\ldots 415.8 = a\ \textit{log.} & 2.618865 & \textit{log.}\, r^t p \\
p = \underline{200} & & \\
\text{the interest} = 215.8l.\ \text{nearly.} & &
\end{array}$$

2. What will 50*l.* amount to in 10 years and 211 days at $4\frac{1}{2}$ *per cent.*?

In this case $r = 1.045$, $p = 50$, $t = 10\frac{211}{365}$.

$$\begin{array}{rrl}
1.045\quad \textit{log.} & 0.019116 & \\
 & 10\frac{211}{365} & \\
\hline
 & 0.202211 & \textit{log.}\, r^t \\
50\ \textit{log.} & 1.698970 & \\
\hline
\text{Amount nearly}\ \overset{l.}{79.649}\quad \textit{log.} & \underline{1.901181} &
\end{array}$$

3. What is the compound interest of 242*l.* 10*s.* forborn $2\frac{1}{2}$ years, at 4 *per cent. per ann.* the interest payable half yearly.

As 100 : 102 :: 1 : 1.02 (the amount of 1 pound in $\frac{1}{2}$ a year) $= r$, $t = 5$ ($\frac{1}{2}$ years), $p = 242.5$.

$$\begin{array}{rrl}
r = 1.02\ldots\ldots\ldots \textit{log.} & 0.008600 \\
t = 5\ldots\ldots\ldots\ldots\ldots & 5 \\
\hline
 & 0.043000 \\
p = 242.5\ldots\ldots\ \textit{log.} & 2.384712 \\
\hline
\text{amount}\quad 267.74\quad \textit{log.} & \underline{2.427712} \\
\underline{242.5} & \\
\text{Interest} = \underline{15.24l.}\ \text{nearly.} &
\end{array}$$

4. What principal will raise a stock of 1000*l*, in 15 years at 5 *per cent.*

Here $r=1.05$, $t=15$, $a=1000$.

$$\begin{array}{rlll} & r=1.05 & \log.\ 0.021189 & \\ & & \underline{\qquad\quad 15} & \\ & & 0.317835 & \log.\ r^{t} \\ & a=1000\ldots\log. & \underline{3.000000} & \\ \textit{Ans,}\ \pounds 481.02=p & \log. & 2.682165 & \text{diff,}\ \log.\ \frac{a}{r^{t}} \end{array}$$

5. At what rate of interest will 480*l*. raise a stock of 864*l*. 8*s*. in 15 years?

Here $a=864.4$, $p=480$, $t=15$.

$$\begin{array}{rll} a=864.4\ldots\log. & 2.936715 & \\ p=480\ldots\ldots\log. & \underline{2.681241} & \\ 15)\ & \underline{0.255474} & \log.\ \frac{a}{p} \\ \text{rate } 1.04\ \log. & \underline{0.017032} & \log.\ \left(\frac{a}{p}\right)^{\frac{1}{t}} \end{array}$$

Ans. 4 *per cent.* nearly.

6. In what time would 575*l*. raise a stock of 756*l*, 14*s*. at 4 *per cent.*?

In this case $r=1.04$, $p=575$, $a=756.7$.

$$\begin{array}{rll} a=756.7\ldots\log. & 2.878924 & \\ p=575\ldots\ldots\log. & \underline{2.759668} & \\ r=1.04\ \log.\ldots 0.017033)\ & \underline{0.119256} & \log.\ \frac{a}{p} \\ & 7 & \end{array}$$

7 quotient nearly, the value of t, or number of years required.

7. In what time would a sum double itself at 5 *per cent.*?

Here if p is the principal, $2p$ is the amount, and the expression $t=\frac{\log.\ \frac{a}{p}}{\log.\ r}$ becomes $t=\frac{\log.\ \frac{2p}{p}}{\log.\ r}$, or $t=\frac{\log.\ 2}{\log.\ r}=\frac{0.301030}{0.021189}=14.2$ *years*, the time nearly.

And the time in which a sum would triple itself, is found by dividing the *log*. of 3 by the *log*. of the *rate*, &c.

Of ANNUITIES.

195. An Annuity, strictly speaking, is a yearly allowance or payment; the term however, is usually applied to any periodical income.

When the annuity is payable immediately, it is said to be in *possession*; but should its commencement depend upon a future event, or not become due till after a certain number of years have elapsed, it is then called an annuity in *reversion*.

If the annuity is not limited in respect of time but supposed to continue for ever, it is called a *perpetuity*.

All the computations relating to annuities are generally made according to compound interest.

Let $r =$ the *rate* or the amount of 1 pound in 1 year, as in compound interest.
$p =$ any annuity, pension, or yearly rent.
$t =$ the time.
$a =$ the amount of the annuity when it is forborn.
$v =$ its value or present worth.

To find the amount (a) in the time t:

The amount of the sum p in t years is pr^{t} (194)
in $t - 1$ years .. pr^{t-1}
in $t - 2$ years .. pr^{t-2}
in $t - 3$ years .. pr^{t-3}
&c. &c.

Therefore the whole amount in t years will be

$$pr^{t} + pr^{t-1} + pr^{t-2} + pr^{t-3} + \ldots . pr^{t-t},$$ or

which is the same thing, $p + pr + pr^{2} + pr^{3} + \ldots pr^{t}$, because $pr^{t-t} = p$, that is, supposing the amount includes the last payment, which bears no interest.

Now (153) $p + pr + pr^2 + \ldots\ldots pr^{t} = p \times \frac{r^{t} - 1}{r - 1} = a$, from which theorem, the following expressions for the several cases of annuities in *arrear* are readily obtained:

$$a = p \times \frac{r^{t} - 1}{r - 1}.$$

$$p = \frac{a\,(r - 1)}{r^{t} - 1}.$$

$$pr^{t} - ar = p - a.$$

$$t = \frac{log.\left(\frac{a\,(r-1)}{p} + 1\right)}{log.\ r}.$$

196. The present worth or value of an annuity (p) supposed to continue t years, is found in the following manner:

Since 1 pound is the present worth of the sum r due at the end of 1 year, we shall have,

$r : 1 :: p : \frac{p}{r}$ the present worth of p pounds due at the end of 1 year; therefore if the sum p becomes due at the end of 2 years, its value at the end of 1 year will also be $\frac{p}{r}$;

whence $r : 1 : \frac{p}{r} : \frac{p}{r^2}$ is the present value of $\frac{p}{r}$ due at the end of 1 year, or the present worth of p due at the end of 2 years:

In like manner we have $\frac{p}{r^3}$ for the present worth of p pounds due at the end of 3 years; hence the present worth of p due at the end of t years will be $\frac{p}{r^{t}}$: consequently $\frac{p}{r} + \frac{p}{r^2} + \frac{p}{r^3} + \ldots \frac{p}{r^{t}}$ (continued to t terms) the sum of all the present worths of the yearly payments, will be the present value of the annuity.

Now this series is a geometrical progression having $\frac{p}{r}$ for the first term, $\frac{1}{r}$ the ratio, and t the number of terms; and its sum (153) is $= \dfrac{\frac{1}{r} \times \frac{p}{r^t} - \frac{p}{r}}{\frac{1}{r} - 1} = \frac{p}{r^t} \times \frac{r^t - 1}{r - 1} = v$.

In the case of a perpetuity, where t or the number of years are supposed to be continued for ever, the last term $\frac{p}{r^t}$ becomes $= 0$, and consequently $\frac{1}{r} \times \frac{p}{r^t} = 0$, and the expression is $\dfrac{-\frac{p}{r}}{\frac{1}{r} - 1}$ or $\frac{p}{r - 1} = v$ the present worth.

From the theorem $v = \frac{p}{r^t} \times \frac{r^t - 1}{r - 1}$, we get the other three expressions which follow:

$$p = v \times \frac{r^{t+1} - r^t}{r^t - 1}.$$

$$t = \frac{\log.\ \dfrac{p}{p + v - vr}}{\log.\ r}.$$

$vr^{t+1} - (p + v)r^t + p = 0$: these four theorems relate to the valuation of annuities.

Examples.

1. If an annuity of 50*l.* be forborn 7 years, what will it amount to at 4 *per cent. per ann.* compound interest?

Here $p = 50$, $r = 1.04$, and $t = 7$; and the expression $p \times \frac{r^t - 1}{r - 1}$ becomes $50 \times \frac{1.04^7 - 1}{.04} = 394.95l. = a$, the amount sought.

2. In how long time will 50*l.* annuity raise a stock of 395*l.* at 4 *per cent. per ann.* compound interest?

In this case $p = 50$, $a = 395$, and $r = 1.04$,

$$\text{and } t = \frac{\log.\left(\frac{a(r-1)}{p} + 1\right)}{\log.\, r} = \frac{\log.\ \frac{395 \times .04}{50} + 1}{\log.\ 1.04} = \frac{\log.\ 1.316}{\log.\ 1.04} = \frac{0.119256}{0.017033} = 7 \text{ years, the required time.}$$

3. If 80*l.* annuity forborn 9 years amounts to 893*l.* what is the rate of interest?

Here $p = 80$, $a = 893$, and $t = 9$; these substituted in the equation $pr^t - ar = p - a$, give $80r^9 - 893r = 80 - 893$, or $r^9 - 11.1625r = -10.1625$:

To approximate the root r by the method of trial-and-error (188) let 1.05 and 1.06 be the first assumptions, because upon trial, its value appears to lie between those numbers:

$$\begin{array}{rr} \text{Then } 1.05^9 - 11.1625 \times 1.05 = & -10.1693 \\ & -10.1625 \\ \textit{error} & .0068 \end{array}$$

$$\begin{array}{rr} 1.06^9 - 11.1625 \times 1.06 = & -10.1427 \\ & -10.1625 \\ \textit{error} & .0198 \\ & .0068 \\ \textit{sum} & .0266 \text{ errors unlike} \end{array}$$

$$\begin{array}{rr} .0198 \times 1.05 = & .020790 \\ .0068 \times 1.06 = & .007208 \\ \text{sum} & .027998 \end{array}$$

$$\frac{.027998}{.0266} = 1.052 \text{ first approximation.}$$

Next, assuming 1.052, and 1.054; and the 2*d.* approximation will be 1.053 which is very nearly the true value of r: hence $1.053 \times 100 = 105.3$, and 5.3*l.* or 5*l.* 6*s.* *per cent.* is the rate required.

4. What is the value of a freehold estate which rents at 50*l.* *per ann.* allowing 5 *per cent.* compound interest?

If the yearly rent is considered as a *perpetuity*, then $p=50$, and $r=1.05$; and the expression $\frac{p}{r-1}$ becomes $\frac{50}{1.05-1}=1000l.$ which is 20 *years* purchase.

5. What is the present worth of 100*l.* annuity to continue 10 years, allowing 5 *per cent. per annum* compound interest, supposing the payments are made quarterly, (viz. 25*l.* every quarter)?

Here $p=25$, $t=40$ (the quarters in 10 years), and $r=1.0125$ the amount of 1 pound in a quarter of a year:

whence $v=\frac{p}{r^t}\times\frac{r^t-1}{r-1}=\frac{25}{1.64362}\times\frac{1.64362-1}{1.0125-1}=783.17l.$ nearly, the value sought.

6. What annuity or yearly income, to continue 20 years, may be purchased for 1000*l.* at $3\frac{1}{2}$ *per cent.*?

In this case $v=1000$, $r=1.035$, $t=20$, whence, by substitution, $p=v\times\frac{r^{t+1}-r^t}{r^t-1}=1000\times\frac{1.035^{21}-1.035^{20}}{1.035^{20}-1}=\frac{69.64}{.98979}=70.36l.$ nearly, the annuity required.

197. To calculate the present value of an annuity in *reversion*, let t denote the *whole* time till it expires (as before), and n the time *before* its commencement:

Then $\frac{p}{r^t}\times\frac{r^t-1}{r-1}-\frac{p}{r^n}\times\frac{r^n-1}{r-1}$, or (by reduction), $\left(\frac{1}{r^n}-\frac{1}{r^t}\right)\times\frac{p}{r-1}$ will evidently be the expression for its present worth. And from this theorem others may be derived for solving the different cases.

ON THE PROPERTIES OF NUMBERS.

198. THE sum of any number of even numbers is an even number.

199. Therefore an even number taken any number of times will make an even number. And consequently the continued product of any number of even numbers will also be even.

200. An even number of odd or of even numbers will be even.

201. The difference of two even, or of two odd numbers will be even.

202 The difference of an even and an odd number will be odd.

203. An odd number taken an odd number of times will make an odd number.

The last six articles may be considered as axioms rather than propositions requiring formal demonstration.

204. If an odd number measures an odd number, the quotient will be odd. This is evident from *art.* 203, because the product of the quotient and divisor is equal to the dividend.

205. If an odd, or an even number measures an even one, the quotient will be even. This follows from *art.* 200.

206. An even number cannot measure an odd number——In other words, if an odd number be divided by an even one, the quotient will always contain a fraction. For an even number taken any *whole* number of times whatever, cannot make an odd number.

207. If one number measures another, it will also measure any multiple of it.

Let d be the measure or divisor, and q the quotient; then dq will be the dividend, and mdq a multiple of it; and $\frac{mdq}{d} = mq$, that is, d measures the number dq, and also its multiple mdq.

208. If a number measures two other numbers, it will also measure their sum, and difference.

Let the measure be d, and a and b the other two numbers; then $\frac{a}{d}$ and $\frac{b}{d}$ are whole numbers (by hypothesis); therefore their sum $\frac{a+b}{a}$, and also their difference $\frac{a-b}{a}$, must be whole numbers.

Corol. 1. Hence, if a number d measures another number $a+b$, and also a part of it b, it will also measure the remaining part a. (Arith. *art.* 40).

Corol. 2. When $a = b$, then the number $a+b$ is an even number (200); therefore, if a number (d) measures an even number ($a+b$), it will also measure its half (a or b).

209. Every number having 0 or 5 in the units place, is divisible by 5.

210. All prime numbers (*i. e.* those which can only be measured by 1) are odd, except the number 2. And such numbers have 1, 3, 7, or 9 in the units place, 2 and 5 excepted. All other numbers are composite or the products of two or more numbers.

211. The least factors of every composite number are its prime divisors. Thus 1, 2, and 3 are the prime divisors of 6, or 12, or 18, &c.

212. The least common multiple of two or more numbers is the continued product of the highest powers of their unlike prime factors.

Let a^2bc, bcd, cd^2f be three numbers, a, b, c, d, f, being the prime factors, all unlike; then a^2d^2bcf is their least common multiple:

For if a^2d^2bcf be divided by either of its factors, the quotient is not divisible by all the three numbers: and whatever number is divisible by those numbers, it must contain a^2, d^2, and the factors bcf, because a, d, b, c, and f are primes, for which reason no number can contain a^2, d^2 and bcf except a^2d^2bcf or some multiple of it, therefore a^2d^2bcf is the least. From this expression, the rule in Arith. *art.* 46, is immediately obtained.

To give an example in numbers, let the least common multiple of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9, be required:

The numbers when resolved into their prime factors will be

$$1,\ 2,\ 3,\ 2^2,\ 5,\ 2 \times 3,\ 7,\ 2^3,\ 3^2,$$

and the continued product of the highest powers of the unlike factors is $1 \times 5 \times 7 \times 2^3 \times 3^2 = 2520$ the multiple required.

The preceding rule is simple. But the great difficulty consists in resolving large numbers into their component factors: nor has any direct method been discovered for that purpose. When a number is composite, one of the factors must be its square root or a less number, and therefore if the number is odd (to which it should be reduced) the odd numbers less than its square root, are the most convenient divisors for resolving it into its factors.

213. The expression $\frac{1 \times 2 \times 3 \times \ldots . (n-1) + 1}{n}$ will give an integer, or a fractional quotient, according as n is a prime, or a composite number: thus if $n = 7$,

$$\text{then } \frac{1 \times 2 \times 3 \times 4 \times 5 \times (7-1) + 1}{7} = \frac{721}{7} = 13.$$

$$\text{If } n = 8, \text{ we have } \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times (8-1) + 1}{8} = 630\tfrac{1}{8}.$$

But if n be a large number, the operation of obtaining the continued product of all the inferior numbers will be too laborious to make the theorem useful in determining whether the number n be prime or composite. See Waring *Meditat. Algeb.* p. 218. and Legendre *Theorie des Nomb.* p. 183.

214. If n be put to denote any of the numbers 1, 2, 3, 4, &c. then $6n + 1$, and $6n - 1$ will give a series containing all the prime numbers greater than 3. But it must be remarked, that neither $6n + 1$ nor $6n - 1$ are *always* prime numbers: Thus if $n = 8$, or 9, then $6n + 1 = 49$, and 55, both composite: or if $n = 6$, we have $6n - 1 = 35$ a composite number. According to Fermat, the expression $2^x + 1$ should always be a prime number if any term of the series 1, 2, 4, 8, 16, 32, 64, &c. be substituted for x: Euler however, has found the theorem defective when $x = 32$, for $2^{32} + 1 = 641 \times 6700417$.

215. If the sum of the digits of a number is divisible by 9, the number itself is also divisible by 9.

Let $a, b, c, d,$ be the digits of a number consisting of 4 figures; then $1000a + 100b + 10c + d$ will express the number:

$$\begin{array}{r l} 9) & 1000a + 100b + 10c + d\ (111a + 11b + c \\ & 999a + 99b + 9c \\ \hline \text{remainder} & a + b + c + d \\ \hline \end{array}$$

Hence it is evident, when the remainder or sum of the digits is divisible by 9, the number itself must be so too, whatever be the number of its figures.

On this property is founded the proof of multiplication by casting away the nines: Arith. *art.* 21.

216. The difference between a number consisting of two digits, and the number formed by the digits when in an inverted order, is always 9 times the difference of the two digits. *Art.* 128. *ex.* 11.

217. The sum of the odd numbers $1 + 3 + 5 + 7 + \ldots . n$ is $= n^2$. (139)

Hence the differences of the squares $1^2, 2^2, 3^2, 4^2$, &c. will be 3, 5, 7, &c.

218. The sum of any number of the series of cubes $1^3+2^3+3^3+4^3$, &c. taken from the beginning, is a square number. (179, *ex.* 3.)

219. The sum of two numbers differing by 1, is equal to the difference of their squares.

Let n and $n+1$ be the numbers: then $2n+1=$ their sum: and $(n+1)^2-n^2=2n+1$.

220. The powers of prime numbers are prime to all numbers except their roots or powers of the roots. This is evident from *art.* 212.

221. If a and b be whole numbers, then $\frac{a^n+b^n}{a+b}$ and $\frac{a^n+b^n}{a-b}$ are both integers when n is an odd number: and $\frac{a^n-b^n}{a+b}$ and $\frac{a^n-b^n}{a-b}$ both integers if n is an even one. (54).

222. If twice a number is the sum of two squares, the number itself is the sum of two squares.

For suppose n to be the number, and let $2n=a^2+c^2$; then $4n=2a^2+2c^2$, and $n=\frac{2a^2+2c^2}{4}=\frac{a^2+2ac+c^2}{4}+\frac{a^2-2ac+c^2}{4}$.

223. The product of the sum of two squares by the sum of two squares, is also the sum of two squares.

For $(a^2+b^2)\times(c^2+d^2)=(db+ac)^2+(ad-bc)^2$.

224. The product of the sum of four squares by the sum of four squares, is the sum of four squares. This theorem has been demonstrated by Euler and Lagrange.

*224. Neither the sum nor difference of two cube numbers is a cube.

225. If n be any prime number, and N any number not divisible by n, then $N^{n-1} - 1$ is divisible by n.

226. If $4n + 1$ be a prime number, it is the sum of two squares. And when $8n + 1$ is a prime number, it is the sum of two, and also of three squares.

227. Every prime number is the sum of four squares.

228. Every number is the sum of four, or of a less number of squares.

> Euler, Lagrange, and others have investigated these latter properties; the demonstrations however, are too long to be admitted in this place.

229. A *perfect number* is equal to the sum of all its aliquot parts.

Thus 6 is a perfect number, its aliquot parts being 1, 2, and 3, whose sum $1 + 2 + 3 = 6$. And 28 is also a perfect number, for $28 = 1 + 2 + 4 + 7 + 14$ the aliquot parts of 28. In the last proposition of Euclid's *9th.* book it is proved, that when the sum of the geometrical series $1 + 2 + 4 + 8 + 16 +$ &c. is a prime number, the said sum multiplied by the last term of the series will be a perfect number. If therefore, n is put to denote the number of terms, $2^n - 1$ will be the sum, and 2^{n-1} the last term; consequently $(2^n - 1) \cdot 2^{n-1}$ is a perfect number when $2^n - 1$ is prime. Thus, if $n = 5$, then $(2^5 - 1)\,2^{5-1} = 31 \times 16 = 496$ the third perfect number.

230. *Amicable numbers*, are pairs of numbers having this property, that each is equal to the sum of all the aliquot parts of the other:

Thus 220 and 284 are amicable numbers; for the sum of 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 which are the aliquot parts of 220, is $= 284$; and 1, 2, 4, 71, 142, the aliquot parts of 284, when added together, make 220: those two numbers are the least of the kind.

The two next amicable numbers are 6232 and 6368 according to Euler, who has treated the subject at very considerable length, and given a table containing 61 pair of these numbers, in a miscellaneous tract published in 1750. In this we are informed that Stifelius was the first who took notice of such numbers.

Many curious investigations relative to the properties of numbers are to be found in Legendre *Essai sur la Théorie des Nombres.*

231. A COLLECTION OF EXERCISES *in the several* RULES *of* ALGEBRA, *beginning with Multiplication.*

1. $(x-a) \times (x+a) = x^2-a^2.$
2. $(5x-7) \times (7x-5) = 35x^2-74x+35.$
3. $(-a-b-1).(a-1) = b-a^2-ab+1.$
4. $(3a^2-2b).(4a+3b) = 12a^3+9a^2b-8ab-6b^2.$
5. $(a^2+2ab+b^2).(a+b) = a^3+3a^2b+3ab^2+b^3.$
6. $(x^2+2xy+y^2).(x-y) = x^3+x^2y-xy^2-y^3.$
7. $(x^3+x^2y+xy^2+y^3).(x-y) = x^4-y^4.$
8. $(x^4-x^3y+x^2y^2-xy^3+y^4).(x+y) = x^5+y^5.$
9. $(\frac{2}{3}x^2y-\frac{4}{7}).(\frac{3}{5}x+\frac{1}{8}) = \frac{2}{5}x^3y+\frac{1}{12}x^2y-\frac{12}{35}x-\frac{1}{14}.$
10. $(\frac{2}{3}a^3-\frac{4}{5}a^2b+\frac{1}{2}b^3):(\frac{3}{5}ab-2b^2) = \frac{2}{5}a^4b-\frac{136}{75}a^3b^2-\frac{8}{5}a^2b^3+\frac{3}{10}ab^4.$
11. $(5a^3b-2ab^3+4a^2c^2).(2a^3b-ab^3+3a^2c^2) = 10a^6b^2-9a^4b^4+23a^5bc^2$
$+2a^2b^6-10a^3b^3c^2+12a^4c^4.$

Division.

Quotients.

1. $x+c)\ x^2-c^2\ (x-c.$
2. $x^2-16)\ x^6-8x^4-124x^2-64\ (x^4+8x^2+4.$
3. $4a^2+5ab+b^2)\ 8a^4-2a^3b-13a^2b^2-3ab^3\ (2a^2-3ab.$
4. $a^2+b^2+c^2)\ a^4+2a^2b^2+b^4-c^4\ (a^2+b^2-c^2.$
5. $\frac{2}{3}a^2-\frac{3}{5}b^2)\ \frac{2}{7}a^3-\frac{9}{35}ab^2+\frac{1}{6}a^2b-\frac{3}{20}b^3\ (\frac{3}{7}a+\frac{1}{4}b.$
6. $a-x)\ a^3-x^3-1\ (a^2+ax+x^2+\frac{-1}{a-x}.$
7. $b-y)\ b^4-3y^4\ (b^3+b^2y+by^2+y^3-\frac{2y^4}{b-y}.$
8. $b^2-5a^2-3ab)\ 13a^3b+19a^2b^2-20a^4-5ab^3\ (4a^2-5ab.$

9. $\dfrac{20a^5-41a^4b+a^3b^2-45a^2b^3+25ab^4-6b^5}{5a^3-4a^2b+5ab^2-3b^3}=4a^2-5ab+2b^2.$

10. $\dfrac{4x^8+32x^7+96x^6+144x^5+132x^4+80x^3+32x^2+8x+1}{2x^4+8x^3+8x^2+4x+1}=2x^4+8x^3$
$+8x^2+4x+1.$

11. $\dfrac{x^6-y^6}{x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5}=x+y.$

The divisors are under the dividends in the three last examples.

Fractions reduced to their lowest terms.

1. $\dfrac{x^2-4xy+4y^2}{x^3-6x^2y+12xy^2-8y^3}$ ………………………… $=\dfrac{1}{x-2y}.$

2. $\dfrac{x^3-b^2x}{x^2+2bx+b^2}$ ………………………… $=\dfrac{x^2-bx}{x+b}.$

3. $\dfrac{a^2-ab-2b^2}{a^2-3ab+2b^2}$ ………………………… $=\dfrac{-a-b}{b-a}.$

4. $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ ………………………… $=\dfrac{a^2+x^2}{a-x}.$

5. $\dfrac{10a^5+20a^4b+10a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$ ………………………… $=\dfrac{10a^4+10a^3b}{a^2b+ab^2+b^3}.$

6. $\dfrac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$ ………………………… $=\dfrac{3a^2+b^2}{4a-b}.$

7. $\dfrac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$ ………………………… $=\dfrac{7a-2b.}{5a^2-3ab+2b^2}.$

Improper fractions reduced to whole or mixt quantities.

1. $\dfrac{12xy-6x^2y^2-x^3y}{8xy}$ ………………………… $=1\frac{1}{2}-\frac{2}{3}xy-\frac{1}{8}x^2.$

2. $\dfrac{x^2-2x+2}{x-1}$ ………………………… $=x-1+\dfrac{1}{x-1}.$

3. $\dfrac{1-x^3}{1-x}$ ………………………… $=1+x+x^2$

4. $\dfrac{1-5x+10x^2-10x^3+5x^4-x^5}{1-2x+x^2}$ ………………………… $=1-3x+3x^2-x^3.$

5. $\dfrac{\frac{1}{2}x^2-\frac{7}{12}xy+\frac{1}{4}y^2}{\frac{2}{3}x-\frac{1}{3}y}$ ………………………… $=\frac{3}{4}x-\frac{3}{4}y.$

6. $\dfrac{a^2-b^2+a-2b}{a-b}$ ………………………… $=a+b+1-\dfrac{b}{a-b}.$

Mixt quantities brought to improper fractions.

1. $4+x+\dfrac{1}{x-4}$ .... $=\dfrac{x^2-15}{x-4}$.

2. $a-b+\dfrac{b}{a+1}$ .... $=\dfrac{a^2-ab+a}{a+1}$.

3. $3x-7+\dfrac{3}{3x+7}$ .... $=\dfrac{9x-46}{3x+7}$.

4. $\frac{1}{3}-\frac{1}{3}x+\dfrac{x-y}{y-x}$ .... $=\dfrac{\frac{1}{3}x^2-\frac{1}{3}xy+\frac{2}{3}x-\frac{2}{3}y}{y-x}$.

5. $x^3+x^2y+xy^2+y^3+\dfrac{y^4}{x-y}$ .... $=\dfrac{x^4}{x-y}$.

6. $1+x+x^2+x^3+x^4+\dfrac{x^5}{1-x}$ .... $=\dfrac{1}{1-x}$.

7. $1+x-x^3-x^4+x^6+\dfrac{x^7-x^8}{1-x+x^2}$ .... $=\dfrac{1}{1-x+x^2}$.

Fractions reduced to common denominators.

1. $\dfrac{x}{2}$, $\dfrac{x}{3}$, and $\dfrac{x}{4}$ .... $=\dfrac{6x}{12}$, $\dfrac{4x}{12}$, and $\dfrac{3x}{12}$.

2. $\dfrac{z}{3}$, $\dfrac{z}{4}$, $\dfrac{z}{5}$, and $\dfrac{z}{6}$ .... $=\dfrac{20z}{60}$, $\dfrac{15z}{60}$, $\dfrac{12z}{60}$, and $\dfrac{10z}{60}$.

3. $\dfrac{n}{a}$, $\dfrac{n}{b}$, and $\dfrac{n}{c}$ .... $=\dfrac{bcn}{abc}$, $\dfrac{acn}{abc}$, and $\dfrac{abn}{abc}$.

4. $\dfrac{a}{bc}$, $\dfrac{b}{ac}$, and $\dfrac{c}{ab}$ .... $=\dfrac{a^2}{abc}$, $\dfrac{b^2}{abc}$, and $\dfrac{c^2}{abc}$.

5. ab, $\dfrac{bc}{a}$, and $\dfrac{cd}{ab}$, ... $=\dfrac{a^2b^2}{ab}$, $\dfrac{b^2c}{ab}$, and $\dfrac{cd}{ab}$.

6. $\dfrac{2x^2-18}{x+3}$, $\dfrac{1}{x-3}$, and $\dfrac{1}{x}$ .... $=\dfrac{2x(x-3)^2}{x^2-3x}$, $\dfrac{x}{x^2-3x}$, and $\dfrac{x-3}{x^2-3x}$.

Addition of fractions.

1. $\dfrac{n}{2}+\dfrac{2n}{3}+\dfrac{3n}{4}+\dfrac{4n}{5}$ .... $=\dfrac{163n}{60}$.

2. $\frac{1}{2}z+\frac{1}{3}z+\frac{5}{6}z+\frac{5}{9}z$ .... $=2\frac{2}{9}z$.

3. $\dfrac{2z}{3}+\dfrac{7z}{4}+\dfrac{2z+1}{3}$ .... $=3\frac{1}{12}z+\frac{1}{3}$.

4. $\frac{a^2+az}{a-z}+a$ .... $=\frac{2a^2}{a-z}$.

5. $\frac{a-b}{a+b}+\frac{a+b}{a-b}+1$ .... $=\frac{3a^2+b^2}{a^2-b^2}$.

6. $a-b+c+\frac{b-a-c}{1+a}$ .... $=\frac{a^2-ab+ac}{1+a}$.

7. $7\frac{1}{2}x+\frac{x-5}{6}+4\frac{1}{2}x+\frac{2x+7}{4}+\frac{3x}{2}$ .... $=14\frac{1}{6}x+\frac{11}{12}$.

8. $\frac{-a}{z}+\frac{-z-b}{5}+\frac{\frac{1}{2}z-1+a}{3}$ .... $=\frac{(5a-3b-5)z-\frac{1}{2}z^2-10a}{15z}$.

9. $\frac{n+z}{n-z}+\frac{z-a}{c-z}$ .... $=\frac{(n+z)\,(c-z)+(n-z)\,(z-a)}{(n-z)\,(c-z)}$.

10. $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4-a^3x}$ .... $=\frac{a}{a-x}$.

11. $\frac{1}{a}-\frac{z}{a^2}+\frac{z^2}{a^3}-\frac{z^3}{a^4}+\frac{z^4}{a^5+a^4z}$ .... $=\frac{1}{a+z}$.

Subtraction of fractions.

1. $\frac{x-7}{8}-\frac{7-x}{6}$ .... $=\frac{7x-49}{24}$.

2. $\frac{x-7}{-8}-\frac{7-x}{-6}$ .... $=\frac{49-7x}{24}$.

3. $\frac{z}{m}-\frac{z}{n}$ .... $=\frac{(n-m)z}{mn}$.

4. $\frac{9x+1}{2}-\left(x-\frac{x-1}{5}\right)$ .... $=\frac{37x+3}{10}$.

5. $\frac{a+c}{a-c}-\frac{a-c}{a+c}$ .... $=\frac{4ac}{a^2-c^2}$.

6. $\frac{a+b}{x}-\frac{a+b}{z}$ .... $=\frac{(a+b)\,(z-x)}{xz}$.

7. $(a+c)\,(a-c)-\frac{-2a^2c^2}{(a+c)(a-c)}$ .... $=a^2+c^2$.

Multiplication of fractions.

1. $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{f}$ .... $=\frac{a}{f}$.

2. $\frac{a^2-c^2}{n}\times\frac{n^2}{a+c}$ .... $=(a-c)n$.

3. $\frac{x+y}{a+b} \times \frac{x-y}{a+b} \cdots\cdots\cdots\cdots = \frac{x^2-y^2}{(a+b)^2}.$

4. $\frac{\frac{1}{2}x-6z}{y} \times \frac{2xy}{3} \cdots\cdots\cdots\cdots = \frac{1}{3}x^2-4xz.$

5. $(3a-6x) \times \frac{3x-1}{a^2-4ax+4x^2} \cdots\cdots\cdots\cdots = \frac{9x-3}{a-2x}$

6. $\frac{5a-x}{4z} \times \frac{-x^2}{1-z} \cdots\cdots\cdots\cdots = \frac{x^3-5ax}{4z-4z^2}.$

7. $\frac{a}{x^2} \times \frac{x^3}{b} \times \frac{x^4}{c} \cdots\cdots\cdots\cdots = \frac{ax^5}{bc}.$

8. $\frac{x^n}{a} \times \frac{x^3}{b} \times \frac{x^2}{c} \cdots\cdots\cdots\cdots = \frac{x^{n+2+3}}{abc}.$

9. $\frac{2x^n}{a(c-b)^2} \times \frac{ax^{-2m}}{d(c-b)^n} \cdots\cdots\cdots\cdots = \frac{2ax^{n-2m}}{ad(c-b)^{n+2}}.$

Division of fractions.

	Divisors.	Dividends.	Quotients.
1.	$\frac{a}{b}$	$\frac{c}{d} \cdots\cdots$	$= \frac{bc}{ad}.$
2.	m	$\frac{x+c}{a+b} \cdots\cdots$	$= \frac{x+c}{m(a+b)}.$
3.	$\frac{x}{a+b}$	$m \cdots\cdots$	$= \frac{m(a+b)}{x}.$
4.	$\frac{a+b}{c}$	$\frac{a^2-b^2}{c^2} \cdots\cdots$	$= \frac{a-b}{c}.$
5.	$\frac{5x}{6}$	$\frac{x+1}{6} \cdots\cdots$	$= \frac{x+1}{5x}.$
6.	$\frac{a^2-b^2}{d}$	$\frac{a^3+b^3}{c} \cdots\cdots$	$= \frac{a^2d-abd+b^2d}{ac-bc}.$
7.	$\frac{a+c}{x-1} \times (n-1)$	$\frac{a-c}{b} \cdots\cdots$	$= \frac{(a-c)(x-1)}{b(a+c)(n-1)}.$
8.	$\frac{2x-2y}{a+b}$	$\frac{8x^3-8y^3}{(a+b)^2} \cdots\cdots$	$= \frac{4x^2+4xy+4y^2}{a+b}.$
9.	$\frac{(a+b)^{-n}}{(c-d)^m}$	$\frac{(a+b)^m}{(c-d)^{-n}} \cdots\cdots$	$= \frac{(a+b)^{m+n}}{(c-d)^{-(m+n)}}.$
10.	$\frac{y^{-m}}{x^{-n}}$	$\frac{x^n}{y^m} \cdots\cdots$	$= \frac{x^0}{y^0} = \frac{1}{1}.$

Fractions resolved into Infinite Series.

1. $\frac{1+x}{1-x}$ .... $=1+2x+2x^2+2x^3+$ &c.

2. $\frac{az}{a-z}$ .... $=z+\frac{z^2}{a}+\frac{z^3}{a^2}+\frac{z^4}{a^3}+$ &c.

3. $\frac{6}{10-1}$ .... $=\frac{6}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+$ &c.

4. $\frac{1}{1-a+a^2}$ .... $=1+a-a^3-a^4+a^6+a^7-a^9$ &c.

5. $\frac{c^2}{(c+x)^2}$ .... $=1-\frac{2x}{c}+\frac{3x^2}{c^2}-\frac{4x^3}{c^3}+$ &c.

6. $\frac{1+x}{(1-x)^2}$ .... $=1+3x+5x^2+7x^3+$ &c.

7. $\frac{a^2+x^2}{a^2-x^2}$ .... $=1+\frac{2x^2}{a^2}+\frac{2x^4}{a^4}+\frac{2x^6}{a^6}+$ &c.

SIMPLE EQUATIONS.

1. Given $\frac{2x}{3}+4=\frac{4x}{5}+12-\frac{5x}{7}$; required x? .. *Ans.* $x=13\frac{47}{61}$.

2. Given $z+\frac{1}{3}z-\frac{1}{7}z+\frac{2z}{5}=7z^2$; req. z? ...... *Ans.* $z=\frac{167}{735}$.

3. Given $\sqrt{x}+\sqrt{(10+x)}=\frac{20}{\sqrt{(10+x)}}$; req. x? *Ans.* $x=3\frac{1}{3}$.

4. Given $x+\sqrt{(a^2+x^2)}=\frac{2a^2}{\sqrt{(a^2+x^2)}}$; req. x? *Ans.* $x=a\sqrt{\frac{1}{3}}$.

5. Given $x^2+a^2=\frac{a^4}{x^2+a^2}$; req. x? .... *Ans.* $x=\sqrt{\left(-\frac{a^2}{2}\right)}$.

6. Given $\frac{ax}{b}+b=\frac{cx}{d}+\frac{ab}{e}$; req. x? .... *Ans.* $x=\frac{(a-c)b^2d}{acd-bc^2}$.

7. Given $\frac{ax}{a-b}+4b=\frac{cx}{3a+b}$; req. x? .. *Ans.* $x=\frac{4b^3+8ab^2-12a^2b}{3a^2+ab-ac+bc}$.

8. Given $\left.\begin{array}{l} ax^2+bx=c \\ dx^2-fx+n=o \end{array}\right\}$ req. x? .... *Ans.* $x=\frac{dc+an}{af+db}$.

9. Given $\left.\begin{array}{l} 5x-3y=24 \\ 11y-7x=14 \end{array}\right\}$ req. x and y? ...... *Ans.* $x=9$, $y=7$.

10. Given $\left.\begin{array}{l} ax+by=n \\ cx+dy=m \end{array}\right\}$ req. x and y? ...... *Ans.* $x=\frac{bm-dn}{bc-da}$.

$$y=\frac{am-cn}{ad-bc}.$$

11. Given $\left.\begin{array}{r} ax+by=d \\ cx-y=x \end{array}\right\}$ req. x and y?.... *Ans.* $x=\frac{d}{bc+a-b}$.

$$y=\frac{cd-d}{bc+a-b}.$$

12. Given $\left.\begin{array}{r} a:x+y::x-y:b \\ x^2+y^2=c \end{array}\right\}$ req. x and y?.. *Ans.* $x=\sqrt{\frac{c+ab}{2}}$.

$$y=\sqrt{\frac{c-ab}{2}}.$$

13. Given $\left.\begin{array}{r} x+y+\frac{y^2}{x}=20 \\ x^2+y^2+\frac{y^4}{x^2}=140 \end{array}\right\}$ req. y?...... *Ans.* $y=6\frac{1}{2}$

14. Given $\left.\begin{array}{r} \frac{1}{2}x+\frac{1}{3}y-\frac{1}{4}z=42 \\ \frac{1}{2}z+\frac{2}{3}y-\frac{1}{5}x=36 \\ \frac{2}{3}z+\frac{2}{3}x-\frac{1}{6}y=47 \end{array}\right\}$ req. x, y, and z? .. *Ans.* $x=60$, $y=54$, $z=24$.

INVOLUTION.

1. What is the square of $ax+bx$? *Ans.* $(a^2+2ab+b^2)x^2$.

2. What is the square of $\frac{1}{2}x-\frac{z}{x}$? *Ans.* $\frac{1}{4}x^2-\frac{z}{x}+\frac{z^2}{x^2}$.

3. Required the cube of $1-\frac{1}{2}x^2$? *Ans.* $1-1\frac{1}{2}x^2+\frac{3}{4}x^4-\frac{1}{8}x^6$.

4. What is the 4*th.* power of $\frac{x}{y}+\frac{y}{x}$? *Ans.* $\frac{x^4}{y^4}+\frac{y^4}{x^4}+\frac{4x^2}{y^2}+\frac{4y^2}{x^2}+6$.

or $x^4y^{-4}+y^4x^{-4}+4x^2y^{-2}+4y^2x^{-2}+6$.

5. What is the square of x^m+y^{-n}? *Ans.* $x^{2m}+2y^{-n}x^m+y^{-2n}$.

or $x^{2m}+\frac{2x^m}{y^n}+\frac{1}{y^{2n}}$.

EVOLUTION *or the Extraction of Roots.*

1. What is the square root of $x^4-4x^3+6x^2-4x+1$?

Ans. x^2-2x+1.

2. Required the square root of $4z^4+12z^3y+13z^2y^2+6zy^3+y^4$?

Ans. $2z^2+3zy+y^2$.

3. What is the square root of $\frac{1}{9}x^2-\frac{1}{2}xy-\frac{4}{9}xz+\frac{9}{16}y^2-yz+\frac{4}{9}z^2$?

Ans. $\frac{1}{3}x+\frac{3}{4}y-\frac{2}{3}z$.

4. What is the square root of a^2+x^2?

Ans. $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$&c.

5. What is the square root of x^2+a^2?

Ans. $x+\frac{a^2}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}-$&c.

6. Required the cube root of $-\frac{27}{125}a^6z^3$? *Ans.* $-\frac{3}{5}a^2z$.

7. What is the cube root of $x^6+6x^5-40x^3+96x-64$?

Ans. x^2+2x-4.

8. What is the cube root of $\frac{1}{8}x^3+\frac{1}{4}x^2y-\frac{3}{4}x^2+\frac{3}{2}x-xy+\frac{1}{6}xy^2+y-\frac{1}{3}y^2+\frac{1}{27}y^3-1$? *Ans.* $\frac{1}{2}x+\frac{1}{3}y-1$.

9. Required the cube root of a^3-b?

Ans. $a-\frac{b}{3a^2}-\frac{b^2}{9a^5}-\frac{5b^3}{81a^8}-\frac{10b^4}{243a^{11}}$ &c.

10. What is the 5*th.* root of a^2-x^2?

Ans. $a^{\frac{2}{5}}\left(1-\frac{x^2}{5a^2}-\frac{2x^4}{25a^4}-\frac{6x^6}{125a^6}, \text{\&c.}\right)$

11. What is the square root of $a+b$?

Ans. $a^{\frac{1}{2}}\left(1+\frac{1}{2}\frac{b}{a}-\frac{1}{8}\frac{b^2}{a^2}+\frac{1}{16}\frac{b^3}{a^3}\text{ \&c.}\right)$

SURDS.

1. Reduce $7\frac{1}{2}$ to the form of the square root? *Ans.* $\sqrt{56\frac{1}{4}}$.

2. Reduce $\frac{1}{3}yz$ to the form of the cube root? *Ans.* $\left(\frac{1}{27}y^3z^3\right)^{\frac{1}{3}}$.

3. Reduce $(a+b)(a-b)$ to the form of the square root?

Ans. $\left(a^4-2a^2b^2+b^4\right)^{\frac{1}{2}}$

4. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to equivalent quantities having a common index?

Ans. $256^{\frac{1}{12}}$ and $125^{\frac{1}{12}}$.

5. Reduce $a^{\frac{2}{3}}$ and $b^{\frac{1}{2}}$ to equivalent quantities having a common index?

Ans. $\left(a^4\right)^{\frac{1}{6}}$ and $\left(b^3\right)^{\frac{1}{6}}$.

6. Let $3^{\frac{1}{3}}$ and $5^{\frac{1}{2}}$ be reduced to equivalent quantities having the common index $\frac{1}{4}$? *Ans.* $\left(81^{\frac{1}{3}}\right)^{\frac{1}{4}}$ and $25^{\frac{1}{4}}$.

7. Reduce $a^{\frac{2}{3}}$ and $b^{\frac{1}{4}}$ to equivalent quantities having the common index $\frac{1}{2}$? *Ans.* $\left(a^{\frac{4}{3}}\right)^{\frac{1}{2}}$ and $\left(b^{\frac{1}{2}}\right)^{\frac{1}{2}}$.

Multiplication of Surds.

1. What is the continued product of $\sqrt{4}$, $\sqrt{5}$, and $\sqrt{7}$? *Ans.* $140^{\frac{1}{2}}$.

2. Required the product of $\sqrt{a^3}$ and $\sqrt{b^3}$? *Ans.* $ab\sqrt{ab}$.

3. What is the product $ax^{\frac{1}{3}}\times bx^{\frac{2}{3}}\times cx^{\frac{1}{2}}$? *Ans.* $abcx^{\frac{3}{2}}$.

4. Required the product $5\times 4^{\frac{1}{3}}\times 18^{\frac{1}{3}}$ *Ans.* $10\times 9^{\frac{1}{3}}$.

5. What is the product $\frac{2}{3}\sqrt{\frac{1}{8}} \times \frac{3}{4}\sqrt{\frac{7}{10}}$? *Ans.* $\frac{1}{40}\sqrt{35}$.

6. What is the product $(a^2b^3)^{\frac{1}{5}} \times (a^3b^2)^{\frac{1}{5}}$? *Ans.* ab.

7. Required the product $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$? *Ans.* $a-b$.

8. Required the product $\sqrt{(2a^2x-8ax+8x)} \times \sqrt{2x}$?

Ans. $2ax-4x$.

9. What is the product $x^2 \times \frac{1}{x^{2\frac{1}{2}}}$ *Ans.* $\frac{1}{x^{\frac{1}{2}}}$ or $x^{-\frac{1}{2}}$.

10. Required the product $(4+2\sqrt{2}) \times (2-\sqrt{2})$? *Ans.* 4.

11. Required the product $x^{\frac{1}{2}} \times (x-b)^{\frac{3}{2}}$? *Ans.* $(x(x-b)^3)^{\frac{1}{2}}$.

Division of Surds.

1. Divide $a^{\frac{n}{m}}$ by $a^{\frac{r}{s}}$? *Quotient* $a^{\frac{ns-mr}{ms}}$.

2. Divide $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$? *Quot.* $x^{\frac{1}{6}}$.

3. Divide $a^{\frac{2}{3}}$ by $b^{\frac{3}{4}}$? *Quot.* $\left(\frac{a^8}{b^9}\right)^{\frac{1}{12}}$.

4. Divide $\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$? *Quot.* $\frac{1}{8}\sqrt{3}$.

5. Divide $\frac{5}{7} \times \left(\frac{2}{3}\right)^{\frac{1}{3}}$ by $\frac{2}{5} \times \left(\frac{3}{4}\right)^{\frac{1}{3}}$ *Quot.* $\frac{25}{21} \times 3^{\frac{1}{3}}$.

6. Divide $z^2-dz-b+d\sqrt{b}$ by $z-\sqrt{b}$? *Quot.* $z+\sqrt{b}-d$.

7. Divide $\sqrt{20}+\sqrt{12}$ by $\sqrt{5}-\sqrt{3}$? *Quot.* $8+2\sqrt{15}$.

8. Divide $(a+z)^{\frac{n}{m}}$ by $(a+z)^{-\frac{r}{s}}$? *Quot.* $(a+z)^{\frac{ns+mr}{ms}}$.

9. Divide $8-5\sqrt{2}$ by $3-2\sqrt{2}$? *Quot.* $4+\sqrt{2}$.

Surds reduced to their simplest terms,

1. Reduce $\sqrt{121b^2x}$ to its most simple terms? *Ans.* $11b\sqrt{x}$.

2. Reduce $875^{\frac{1}{3}}$ to its simplest terms? *Ans.* $5 \times 7^{\frac{1}{3}}$.

3. Reduce $\left(\frac{16}{81}\right)^{\frac{1}{3}}$ to its most simple terms? *Ans.* $\frac{2}{9} \times 18^{\frac{1}{3}}$.

4. Reduce $(z^4-a^2z^3)^{\frac{1}{3}}$ to its most simple terms? *Ans.* $z(z-a^2)^{\frac{1}{3}}$.

5. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to more simple terms? *Ans.* $\sqrt{5}+\sqrt{2}$.

6. Let $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$ be reduced to its most simple terms?

Ans. $8+2\sqrt{15}$.

Addition and Subtraction of Surds.

1. What is the sum of $500^{\frac{1}{3}}$ and $108^{\frac{1}{3}}$? *Ans.* $8 \times 4^{\frac{1}{3}}$.

2. .... of $27^{\frac{1}{2}}$ and $147^{\frac{1}{2}}$? *Ans.* $10\sqrt{3}$.

3. .... of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$? *Ans.* $(2a+24x^2)\sqrt{b}$

4. .... of $9\sqrt{243}$ and $10\sqrt{363}$? *Ans.* $191\sqrt{3}$.

5. .... of $\sqrt{27a^4x}$ and $\sqrt{3a^2x}$? *Ans.* $(3a^2+a)\sqrt{3x}$.

6. .... of $5^{\frac{1}{2}}$ and $5^{\frac{1}{3}}$? *Ans.* $(5^{\frac{1}{6}}+1)\,5^{\frac{1}{3}}$.

7. What is the difference of $\sqrt{448}$ and $\sqrt{112}$? *Ans.* $4\sqrt{7}$.

8. .... of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$?
Ans. $(4a^2 \backsim 2ax)\sqrt{5x}$.

9. .... of $8\,(a^3b)^{\frac{1}{3}}$ and $(a^6b)^{\frac{1}{3}}$?
Ans. $(8a \backsim a^2)\,b^{\frac{1}{3}}$.

10. .... of $\left(\frac{2}{3}\right)^{\frac{1}{3}}$ and $\left(\frac{9}{32}\right)^{\frac{1}{3}}$? *Ans.* $\frac{1}{12} \times 18^{\frac{1}{3}}$.

Powers, and Roots of Surds.

1. What is the square of $ax^{\frac{n}{m}}$? *Ans.* $a^2x^{\frac{2n}{m}}$.

2. Required the cube of $4^{\frac{1}{3}}x^{\frac{2}{3}}$? *Ans.* $4x^2$.

3. What is the $\frac{n}{m}$th. power of $(a+b)^{\frac{r}{s}}$? *Ans* $(a+b)^{\frac{rn}{sm}}$.

4. What is the square of $5-\sqrt{5}$? *Ans.* $30-10\sqrt{5}$.

5. What is the cube of $3z-2\sqrt{x}$?
Ans. $27z^3-54z^2\sqrt{x}+36zx-8x\sqrt{x}$.

6. Let $\frac{ax^2}{bx^{-3}}$ be raised to the nth. power? *Ans.* $\frac{a^n}{b^n}x^{5n}$.

7. Required the 4th. power of $\frac{1}{a}\sqrt{a}$? *Ans.* $\frac{1}{a^2}$.

8. What is the nth. root of a^m? *Ans.* $a^{\frac{m}{n}}$.

9. Required the $\frac{n}{m}$ root of $x^{\frac{r}{s}}$? *Ans.* $x^{\frac{mr}{ns}}$.

10. What is the square root of $9x-6a\sqrt{x}+a^2$? *Ans.* $3\sqrt{x}-a$.

11. What is the square root of $1\frac{7}{9}-2\frac{2}{3}\sqrt{\frac{1}{3}}$? *Ans.* $\frac{2}{3}-2\sqrt{\frac{1}{3}}$.

12. Required the square root of $12-\sqrt{140}$? *Ans.* $\sqrt{7}-\sqrt{5}$.

13. What is the $-n$th. root of $(x+y)^{-m}$? *Ans* $(x+y)^{\frac{m}{n}}$.

Questions producing SIMPLE EQUATIONS.

1. The difference of two numbers being $\frac{1}{4}$, and the difference of their squares 2, then what are the numbers?

Ans. $3\frac{1}{2}$ and $3\frac{3}{4}$.

2. The whole number of troops in two companies are 180, and the number in one troop to the number in the other as 8 to 7. What is the strength of each? *Ans.* 96, and 84 *men.*

3. What two fractions are those whose sum is 1, and the greater divided by the less gives the quotient 20?

Ans. $\frac{20}{21}$ and $\frac{1}{21}$.

4. A General having detached 400 men to take possession of a strong post, and $\frac{4}{9}$ of the remainder of his troops to watch the motions of the enemy, finds that he has only $\frac{5}{17}$ of his army left; what was his whole force? *Ans.* 850 *men.*

5. Three battalions of unequal force are in column of march; the extent of the first battalion is 210 *paces,* the extent of the second is equal to that of the first and $\frac{3}{4}$ of the third together, and the extent of the third is equal to that of the first and half the second; what is the length of the column?

Ans. 1302 *paces.*

6. A company of foot are 1165 of their own paces a head of a troop of horse; now if the foot take 5 paces to every 4 of the horse, but 3 paces of a horse are equal in extent to 4 paces of the foot; how many paces will the horse have marched before they overtake the foot? *Ans.* 13980.

7. If a person buys a certain number of eggs at 2 for a penny, and the like number at 3 a penny, and by selling the whole together at 5 for 2 pence, loses 1 penny; what was the number bought? *Ans.* 30.

8. If the agents A and B acting separately, produce a like effect *a* in the times *b* and *c*, respectively, and A, B, and C

together produce the same effect (a) in the time d; in what time would C alone produce the effect m?

Ans. $\frac{bcdm}{abc - adc - adb}$.

9. A labourer agreed to serve 10 weeks upon these conditions, that for every day he worked he was to receive 2*s.* 4*d*, but to forfeit 7*d.* for every day he absented himself; now at the end of the time he had to receive 4*l.* 19*s.* 2*d.* What number of days did he work? *Ans.* 46.

10. The weight of a cubic foot of *copper* is 9000
of *tin* 7320
of *gun metal* 8784 } *ounces.*

Those numbers also denote the *specific gravities* of the metals: hence the quantity of copper and of tin in the mixture which is gun metal, is required?

Ans. $7842\frac{6}{7}$ ounces of *copper*,
$941\frac{1}{7}$ ounces of *tin*; or $8\frac{1}{3}$*lb.* of *copper* to 1 of *tin*, nearly.

11. Suppose the weight of a brass 12 pounder is 18 *hundred weight*, and that of another brass 12 pounder exactly of the same dimensions is 16 *hundred weight*; now if the former is gun metal whose specific gravity is 8784, it is required to find the weight of *copper* and also of *tin* in the latter piece?

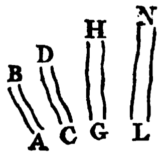
Oz.
Ans. 9600 *copper*
19072 *tin.*

12. Suppose 25 battalions of troops have to march through 4 roads or defiles,

viz.

AB = 1 mile, in this the length of a battalion on the march is 243 paces of $2\frac{1}{2}$ *feet* each, and the rate of marching 65 paces per minute.

CD = $1\frac{1}{4}$ miles, in this a battalion extends 210 paces, and the rate of marching is 50 paces per minute.



GH = $1\frac{1}{2}$ miles, here a battalion is 204 paces in length, and the rate of marching 60 paces per minute.

LN = 2 miles, in this the extent of a battalion is 232 paces, and the rate of marching 80 paces per minute.

Now how should the 25 battalions be divided that the whole march through the 4 roads may be made in the least time, supposing the 4 divisions begin the march together at A, C, G, and L, respectively? *Ans.* 10 battal. through AB.
4 through CD.
5 through GH.
6 through LN.

13. To divide a given number n into two such parts that the difference of their squares shall be equal to a given number d?

Ans. the two parts are $\frac{n^2+d}{2n}$, and $\frac{n^2-d}{2n}$.

14. A body of 1905 troops consists of three battalions; now $\frac{1}{2}$ the first battalion is to $\frac{1}{3}$ of the second, as 7 to 5; and $\frac{3}{5}$ of the second battalion is to $\frac{3}{4}$ of the third as 9 to 10. Required the strength of each battalion? *Ans.* 630,675,600 *men.*

15. A waterman finds that he can row 5 miles *with* the tide in $\frac{3}{4}$ of an hour, and that it takes him $1\frac{1}{2}$ hours to row the same distance back *against* the tide when it is but $\frac{1}{2}$ as strong; hence the velocity of the strongest tide is required?

Ans. $2\frac{2}{9}$ *miles per hour.*

16. A Garrison had provisions sufficient for 30 months, but at the end of 4 months the number of troops were doubled, and 3 months after that it was reinforced with 400 men more, by which means the provisions lasted but 15 months in the whole. Required the strength of the garrison before any augmentation took place? *Ans.* 800 *men.*

17. The weight of a cubic foot of rain water is 1000 *ounces avoirdupois,* and that of a cubic foot of sea water 1031 *ounces*; now how much of each must be taken that a cubic foot of the mixture shall weigh 1008 *ounces*? *Oz.*

Ans. 741 $\frac{29}{31}$ rain water.
266 $\frac{2}{31}$ sea water.

18. Suppose three ingots of metal composed of gold, silver, and copper, each weighing 16 ounces; the first contains 7 *ounces* of gold, 8 of silver, and 1 of copper; the second 5 *ounces* of gold, 7 of silver, and 4 of copper; and the third 2 *ounces* of gold, 9 of silver, and 5 of copper; now what quantity of each ingot must be taken to make another mixture of 16 *ounces* that shall contain $4\frac{15}{16}$ *ounces* of gold, $7\frac{10}{16}$ of silver, and $3\frac{7}{16}$ of copper? *Ans.* 4 *ounces* of the first ingot, 9 of the second, and 3 of the third.

19. If 11520 be divided into 3 parts such, that the sum of the first and second is to the sum of the second and third as 7 is to 9; and the difference of the first and second is to the sum of the first and third, as 1 to 8, the three results will be the number of cartridges for three companies of foot, respectively, 40 rounds to each man. Hence the strength of each company is required? *Ans.* 72, 96, and 120, *men*.

20. The number of men in three companies of foot are such, that the first company with $\frac{1}{2}$ the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, are the same, each being 119 men. Hence the respective numbers are required? *Ans.* 35 men in the first company.
77 in the second.
91 in the third.

21. A man dying, his wife being with child, ordered by will, that if the child proved a daughter, then his wife should have $\frac{2}{3}$ and the child $\frac{1}{3}$ of his estate; but if it was a son, then he should have $\frac{2}{3}$ and the mother $\frac{1}{3}$ thereof; now it happened that the mother was delivered of a son and a daughter; how must the estate, which was 6300*l.* be divided to answer the father's intention?

	l.	
Ans.	900	the daughter's share.
	1800	the mother's.
	3600	the son's.

22. Suppose 4 footmen were to start together to travel the same way round an island which is 124 miles in circumference, and that the first went 11 miles *per day*, the second 15, the third 19, and the fourth 23: when would they come together again? *Ans.* in 31 *days*.

23. Several detachments of Artillery divided a certain number of cannon shot in the following manner:

The first detachment took 72 and $\frac{1}{9}$ of the remainder.
The second took 144 and $\frac{1}{9}$ of the remainder.
The third took 216 and $\frac{1}{9}$ of those that were left.
The fourth took 288 and $\frac{1}{9}$ of those left; and so on.

Now at last it was found that the shot had been equally divided. Hence the whole number of balls, and the number of detachments are required? *Ans.* No. of shot 4608.
Detachments 8.

QUADRATIC EQUATIONS.

1. Given $x^2 - x - 400 = 1700$; to find x.
Ans. $x = (2100\frac{1}{4})^{1} + \frac{1}{2}$.

2. Given $9x^2 + 6x - 27 = 228$; to find x. *Ans.* $x = 5$.

3. Given $ax^2 + x = b$; required x. *Ans.* $x = \frac{1}{2a}(4ab + 1)^{\frac{1}{2}} - \frac{1}{2a}$.

4. Given $x - \sqrt{x} = b$; required x. *Ans.* $x = (\frac{1}{2} \pm (b + \frac{1}{4})^{\frac{1}{2}})^{2}$.

5. Given $ax^{n} - bx^{\frac{n}{2}} - c = -d$; to find x.
Ans. $x = \left(\frac{b}{2a} \pm \left(\frac{4ac - 4ad + b^2}{4a^2}\right)^{\frac{1}{2}}\right)^{\frac{2}{n}}$.

6. Given $x + \sqrt{(5x + 10)} = 8$; required x. *Ans.* $x = 3$.

7. Given $x - \frac{x - y}{2} = 4$

$y - \frac{x + 3y}{x + 2} = 1$; required x and y. *Ans.* $x = 2$ or 5.
$y = 6$ or 3.

8. Given $xy = 125x + 300y$
$y^2 - x^2 = 90000$; required x and y.
Ans. $x = 400$, $y = 500$.

Questions producing QUADRATIC EQUATIONS.

1. To find a number such, that if you subtract it from 30, and multiply the remainder by the number itself, the product shall be 209? *Ans.* 11 or 19.

2. The difference of two numbers is 5, and the difference of their cubes is 1685; what are those numbers? *Ans.* 8 and 13.

3. When 962 men were drawn up in two square columns (i. e. the number of ranks equal to the number of men in front) it was found that one column consisted of 18 ranks more than the other: hence the strength of each column is required? *Ans.* 841, and 121 *men.*

4. To find two numbers whose product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes? *Ans.* $\frac{1}{2}\sqrt{5}$, and $\frac{5+\sqrt{5}}{4}$.

5. Two partners A and B gained 140*l.* by trade; A's money was 3 months in trade, and his gain was 60*l.* less than his stock; and B's money, which was 50*l.* more than A's, was in trade 5 months; what was A's stock? *Ans.* 100*l.*

6. A and B take, in trade, 5940*l.* *per annum* each, but A, whose profits are 2 *per cent.* greater than those of B, clears 100*l.* *per annum* more than B. What are the profits of each, *per cent.* and what do they clear *per annum?*

Ans. A gains 10 *per cent.* and clears 540*l.* *per ann.*
B gains 8 *per cent.* and clears 440*l.* *per ann.*

7. What two numbers are those whose difference multiplied by the difference of their squares will produce 576; and whose sum multiplied by the sum of their squares is 2336? *Ans.* 11 and 5.

8. What number is that which being multiplied by the sum of its two digits, the product shall be 1012, and if 63 be subtracted from the number, its digits will be inverted?

Ans. 92.

9. When 732 men were drawn up in column, the number in front and the number of ranks together made 73. How many were the rank? *Ans.* 61 or 12.

10. Two detachments of foot are ordered to a station distant 39 miles, they begin their march at the same time, but one party by travelling $\frac{1}{4}$ of a mile an hour more than the other, arrives 1 hour sooner: hence the rates of marching are required?

Ans. $3\frac{1}{4}$, and 3 miles *per hour.*

11. To find two numbers whose product shall be 320, and the difference of their cubes to the cube of their difference, as 61 is to 1? *Ans.* 20 and 16.

12. Given the sum of three numbers in harmonic proportion $= 191$, and the product of the first and third $= 4032$; to find the numbers? *Ans.* 72, 63, 56.

13. Given the sum of 3 numbers in geometrical progression $= 91$, and the sum of their squares $= 4459$; what are the numbers? *Ans.* 7, 21, 63.

14. If the sum of two numbers is 11, and the sum of their 5*th.* powers 17831; what are the numbers? *Ans.* 4 and 7.

INDETERMINATE PROBLEMS.

1. To find the least whole number which being divided by 17 shall leave a remainder of 7, but when divided by 26 the remainder shall be 13? *Ans.* 143.

2. Required the least possible integer that being divided by 28, 19, and 15, the respective remainders shall be 19, 15, and 11? *Ans.* 7691.

3. When a company of foot was drawn up in column with 11 men in front, it was found that 5 men were wanting to form complete ranks; but when they were drawn up with 7 men in front, only 1 was required; what was the strength of the company, supposing the number less than 100? *Ans.* 83 *men.*

4. To find the year when the Roman Indiction was 4, the Golden Number 2, and Cycle of the Sun 12? *Ans.* in 1711.

5. A regiment of foot (less than 1000) when put in column with 13 men in front, wanted 9 men to complete the last rank; when 15 were in front then 14 men were wanting; but with 17 in front the ranks were complete: what was the strength of the regiment? *Ans.* 901 *men.*

6. How many different ways is it possible to pay 20*l.* without any other coin than *half guineas* and *half crowns*? *Ans.* 7.

7. If $17x + 19y + 21z = 400$; how many positive integral values are there of x, y, and z? *Ans.* 10 of each.

8. To find two whole numbers having 77 for the difference of their squares? *Ans.* 2 and 9, or 38 and 39.

9. To find that number which being any how divided into two unequal parts, the greater part added to the square of the less, shall be equal to the less part added to the square of the greater? *Ans.* 1.

10. To find the two least whole numbers whose difference, the difference of their squares, and the difference of their cubes, are all square numbers? *Ans.* 6 and 10.

11. To find a square number to which if you add 7, or subtract 7, the sum, and difference, shall also be square numbers? *Ans.* $7\frac{12769}{14400}$.

12. To divide 10 into 4 such parts, that the sum of every three shall be a square? *Ans.* 6, 1, $\frac{6906}{4225}$, and $\frac{5769}{4225}$.

13. To divide $\frac{1}{4}$ into 4 parts such, that either part when added to the cube of $\frac{1}{4}$ shall be a square? *Ans.* $\frac{16790}{400000}$, $\frac{18750}{400000}$, $\frac{29750}{400000}$, and $\frac{34710}{400000}$.

ARITHMETICAL PROGRESSIONS.

1. If the first term $= \frac{1}{2}$, number of terms $= 20$, and the sum of all the terms $= 100$; what is the common difference? *Ans.* $\frac{1}{2}$.

2. A detachment of foot have to occupy a post distant 159 miles; the first day they march 16 miles, the second day $15\frac{1}{2}$, the third day 15, and so on, lessening each day's march $\frac{1}{2}$ a mile: in what time will the journey be performed? *Ans.* 12 *days.*

3. A detachment marched 198 miles in 16 days, and the first day they travelled 18 miles; now supposing each day's march was diminished by the same distance, how far did they travel the last day? *Ans.* $6\frac{3}{4}$ *miles.*

4. If the first term of a progression is $= 0$, common difference $= 1\frac{1}{2}$, and sum of all the terms $= 1170$; what is the number of terms? *Ans.* 40.

5. A party of foot begin their march at 6 in the morning, and travel $3\frac{1}{2}$ miles an hour; 3 hours after a troop of horse follow them from the same place, and march $3\frac{1}{2}$ miles the first hour, 4 miles the next, $4\frac{1}{2}$ the third, &c. increasing their march $\frac{1}{2}$ a mile every hour; in what time will they overtake the foot? *Ans.* 7 *hours.*

6. Given the sum of the squares of the two means $= 346$, and the sum of the squares of the two extremes $= 410$; to determine the four numbers. *Ans.* 7, 11, 15, 19.

7. If a complete square pile of cannon balls contains just $7\frac{7}{40}$ times the number in the bottom layer; then how many are there in the pile? *Ans.* 2870.

8. The cannon shot of a complete triangular pile when placed in rows that touched one another on the ground, formed an exact square. What was the whole number of balls, the number being greater than 4?

GEOMETRICAL PROGRESSIONS.

1. What is the sum of the first 11 terms of the series, 9, $4\frac{1}{2}$, $2\frac{1}{4}$, $1\frac{1}{8}$, &c. *Ans.* $13\frac{1017}{1024}$.

2. What is the $13th.$ term of the progression 21, 7, $2\frac{1}{3}$, $\frac{7}{9}$, &c. *Ans.* $\frac{7}{177147}$.

3. Required the sum of the progression a, $\frac{a}{r}$, $\frac{a}{r^2}$, $\frac{a}{r^3}$, &c. infinitely continued, r being greater than 1? *Ans.* $\frac{ar}{r-1}$.

4. If the first term $= 6$, the ratio $= \frac{1}{2}$, and sum of the progression $= 12$? what is the number of terms?

5. There are 4 numbers in geometrical progression, and the sum of the two least $= 20$, and that of the two greatest $= 45$; what are the numbers? *Ans.* 8, 12, 18, 27.

6. To find 4 numbers in arithmetical progression which being increased by 4, 7, 48, and 224, respectively, the sums shall be in geometrical progression? *Ans.* 7, 29, 51, 73.

7. From a vessel containing 10 gallons of brandy, 1 gallon was drawn out, and a gallon of water poured into the vessel; a gallon of the mixture was then drawn out, and another gallon of water poured in; now the like process being repeated 10

times, it is required to find how much brandy remained in the vessel, supposing the two fluids were thoroughly mixed every time? *Ans.* $3\frac{486784411}{1000000000}$ *gall.*

8. The sum of three numbers in geometrical progression is 91, and their continued product 9261; what are the numbers? *Ans.* 7, 21, 63.

9. If the first term of a series be 90, the last term 2, and the number of terms 20; what is the ratio? *Ans.* .8184438, nearly.

10. Suppose the first term is 1, the last 0, and the sum of the series $\frac{3}{3-x}$; what is the ratio? *Ans.* $\frac{x}{3}$.

PERMUTATIONS, COMBINATIONS, &c.

1. How many changes or variations can take place in the letters of the word *change*? *Ans.* 720.

2. Suppose 7 men stand in a rank; how many times can their order be varied? *Ans.* 5040.

3. If a company consisting of 30 men are drawn up in column, with how many different fronts can that be done, when 5 men are always in front? *Ans.* 142506.

4. How many different hands can be held at the game of whist? *Ans.* 635013559600.

5. How many variations may be made of the letters in the word *Bacchanalia*? *Ans.* 831600.

6. How many different numbers can be made out of an unit, 2 twos, 3 threes, 4 fours, and five fives, taken 5 at a time? *Ans.* 2111.

7. How many different numbers can be made with the same figures as in the last example, supposing all the 15 figures to be in every number? *Ans.* 37837800.

8. Let there be 5 ranks of men, and suppose the first rank consists of 7 men, the second of 10, the third of 12, the fourth of 14, and the fifth of 15; now how many ways can 5 men be chosen from the ranks, one man being taken from each rank every time? *Ans.* 176400.

9. How many words, significant and insignificant, can be made out of the 24 letters?

Ans. 1391724288887252999425128493402200.

Recurring, and other SERIES.—*Differential Method.*

1. What is the sum of the infinite series $\frac{3}{4}-\frac{9}{16}+\frac{27}{64}-\frac{81}{256}+$ &c.

Ans. $\frac{3}{7}$.

2. Required the sum of the series $1+3x+6x^2+10x^3+15x^4+$ &c. infinitely continued?

Ans. $\frac{1}{(1-x)^3}$.

3. What is the sum of the infinite series $1+4x+9x^2+16x^3+$ &c.?

Ans. $\frac{1+x}{(1-x)^3}$.

4. What is the sum of the infinite series $\frac{x}{a}-\frac{x^2}{a^2}+\frac{x^3}{a^3}-\frac{x^4}{a^4}+$ &c.?

Ans $\frac{x}{a+x}$.

5. Required the sum of the infinite series $\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+$ &c.

Ans. $\frac{3}{4}$.

6. What is the sum of the first 10 terms of the series $1+\frac{2}{3}+\frac{3}{9}+\frac{4}{27}+\frac{5}{81}$ &c. *Ans.* $2\frac{4915}{19683}$.

7. Required the sum of the series $1 + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} +$ &c. *in infin.*

Ans. $\frac{4}{3}$.

8. Required the sum of the infinite series $\frac{3}{5} + \frac{3.4}{5.6} + \frac{3.4.5}{5.6.7} + \frac{3.4.5.6}{5.6.7.8}$ &c.

Ans. 3.

9. What is the sum of $\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6}$ &c. infinitely continued?

Ans. $\frac{1}{18}$.

10. Required the sum of the infinite series $\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}$ &c.

Ans. 1.

11. Required the sum of 20 terms of the series of products $1.2.3 + 3.4.5 + 5.6.7 +$ &c. *Ans.* 352380.

12. If the top row of a complete oblong pile of cannon shot consists of 10 balls, and the number of courses are 11; then how many shot are in the pile? *Ans.* 1100.

13. The sides of the top course of a broken rectangular pile of shot are 12 and 7, and the number of courses 9; required the number in the pile?

Ans. 1728.

14. What is the 20*th*. term of the series 1, 6, 21, 56, 126, 252, &c.?

Ans. 42504.

Reversion of Series.

1. To revert the series $ay + by^2 + cy^3 + dy^4 + fy^5$, &c $= x$. (180)

$$Ans.\ y = \frac{x}{a} - \frac{bx^2}{a^3} + \frac{(2b^2 - ac)\,x^3}{a^5} - \frac{(5b^3 - 5abc + a^2d)\,x^4}{a^7}$$

$$+ \frac{(14b^2 - 21ab^2c + 6a^2bd + 3a^2c^2 - a^3f)\,x^5}{x^9} - \text{\&c.}$$

2. To revert the series $ay + \frac{a^2y^2}{2} + \frac{a^3y^3}{2.3} + \frac{a^4y^4}{2.3.4} + \frac{a^5y^5}{2.3.4.5} +$ &c. $= x$.

$$Ans.\ y = \frac{x}{a} - \frac{x^2}{2a} + \frac{x^3}{3a} - \frac{x^4}{4a} + \frac{x^5}{5a} - \text{\&c.}$$

3. To revert the series $y - 2y^2 + \frac{8}{3}y^3 - \frac{13}{3}y^4 +$ &c. $= x$.

$$Ans.\ y = x - 2x^2 + \frac{16}{3}x^3 + \frac{53}{3}x^4 \text{ \&c.}$$

4. Let the series $y+\frac{y^3}{6a^2}+\frac{3y^5}{40a^4}+\frac{5y^7}{112a^6}+\&c.=x$, be reverted.

Ans. $y=x-\frac{x^3}{6a^2}+\frac{x^5}{120a^4}-\frac{x^7}{5040a^6}+\&c.$

5. To revert the series $y^{\frac{1}{2}}+by^4+cy^{7\frac{1}{2}}+dy^{11}+\&c.=x$.

Ans. $y=x^2-2bx^9+(17b^2-2c)x^{16}+(48bc-200b^3-2d)x^{23}$ &c.

Cubic, *and higher* Equations.

1. If $2x^3-24x^2+96x=378$; what is the value of x? *Ans.* $x=9$.

2. Let $x^3+9x=1430$; required the value of x? *Ans.* $x=11$.

3. Given $x^3+21x^2-196x=4116$; to find x?

Ans. $x=14, -14, -21$, the three roots.

4. Given $x^3+7x^2-43x=301$; to find x?

Ans. $x=-7, \sqrt{43}, -\sqrt{43}$, the three roots.

5. Suppose $x^3-171.91x^2+7905.6x=71256$; required the value of x?

Ans. $x=11.862, 60.106, 99.942$, the three roots, nearly.

6. The sum of 4 numbers in geometrical progression being 140, and their continued product $=109395\frac{9}{16}$; what are the numbers?

Ans. $3\frac{1}{2}$, $10\frac{1}{2}$, $31\frac{1}{2}$, $94\frac{1}{2}$.

7. The sum of 3 numbers in harmonic proportion is 191, and their continued product 254016. Required the numbers.

Ans. 72, 63, and 56.

8. Given the sum of three numbers $=32$, the sum of their squares $=350$, and the sum of their cubes $=3926$. What are the numbers?

Ans. 9, 10, and 13.

9. A company of foot can be drawn up in column with 34220 different fronts having always 3 men in front: what is its strength?

Ans. 60 *men.*

10. The number of cannon shot in a complete triangular pile is 9139; then how many are in the bottom course? *Ans.* 703.

11. The number of cannon shot in a complete square pile or pyramid exceeds the number in a complete triangular one by 2300 when the sides of the two bases are equal: how many balls are in each pile?

Ans. 4900 and 2600.

12. If $\frac{x}{a}-\frac{x^2}{a^2}+\frac{x^3}{a^3}-\frac{x^4}{a^4}+\&c.$ *in infin.* $=m$; what is the value of x?

Ans. $x=\frac{ma}{1-m}$.

13. Suppose $1+3x+6x^2+10x^3+15x^4+$ &c. *in infin.* $=10$; required the value of x?

Ans. $x=1-\frac{1}{10^{\frac{1}{3}}}$.

14. If the sum of the series of biquadrates $1^4+2^4+3^4+4^4+$ &c. be equal to 6867 times the number of terms; what is the sum of the series?

Ans. 89271.

15. If $x^4-.611977x^3+.755698x^2-.376366x=.26406285$. Required the value of x? *Ans.* $x=.791207$.

16. Suppose $3^{2x}+3^{x}=4785156$: what is the value of x?

Ans. 7.

17. If $2^{x}-2^{\frac{1}{2}x}=12$; required the value of x?

Ans. 2.33985 *nearly.*

18. Given $1+2x^2+3x^4+4x^6+$ &c. (*in infin.*) $=y^4$

and $2x=y$.

Required the values of x and y?

Ans. $x=\sqrt{\frac{1}{2}}$.

$y=2\sqrt{\frac{1}{2}}$.

19. Suppose $\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+$ &c. (*in infin.*) $=\frac{1}{4}$; what is the value of x?

Ans. $x=\frac{1}{2}-\frac{1}{16}+\frac{1}{96}-\frac{1}{512}+\frac{1}{2560}-$ &c.

20. Given $x^4+y^4=10000$, and $x^5-y^5=25000$; to find x and y?

Ans. $x=8.87047$, and $y=7.85585$ *nearly.*

INTEREST, *and* ANNUITIES.

1. A sum of money put out at simple interest amounts to 297*l.* 12*s.* in 8 months; and the amount of the same sum in 15 months is 306*l.* Required that sum: also the rate of interest? *Ans.* 288*l.* the sum.

5 *per cent.* the interest.

2. Two notes, one of 120*l.* payable in 6 months, and the other of 150*l.* payable in 9 months, were discounted for 8*l.* 10*s.* what was the rate of interest?

Ans. 5*l.* 1*s.* 10½*d.* *per cent.* nearly

3. There is 320*l*. due to me at this time, and 96*l*. more will be due at the end of 5 years (both from the same person); now we make an agreement that the whole shall be discharged at one payment, at the time when the interest of the 320*l*. becomes equal to the discount of the 96*l*. Hence the time of payment is required: the calculation being made at 5 *per cent. per ann.* simple interest? *Ans.* at the end of 1 year.

4. At what rate of compound interest will 481*l*. raise a stock of 1000*l*. in 15 years? *Ans.* 5 *per cent.*

5. What is the amount of 217*l*. forborn 2¼ years at 5 *per cent. per annum* compound interest, supposing the interest payable quarterly? *Ans.* 241*l*. 13*s*. 4½*d*.

6. If 356*l*. be payable at the end of 7 years, what is it worth in ready money, discompting after the rate of 7 *per cent. per ann.* compound interest? *Ans.* 221*l*. 14*s*. *nearly*.

7. The compound interest of a certain sum of money amounted to 344.81*l*. in 4 years; but the simple interest of the same sum, at the same rate in 4 years, would have been only 320*l*. Hence the principal, and the rate of interest are required?

Ans. 1600*l*. the principal.
5 *per cent.* the rate.

8. What is the present worth of an annuity or rent of 50*l*. *per ann.* payable yearly for 21 years, reckoning compound interest at the rate of 6 *per cent. per annum.*

Ans. 588*l*. 4*s*. ¾*d*. *nearly*.

9. For how long time will 600*l*. purchase an annuity of 100*l*. at 4 *per cent.* compound interest? *Ans.* 7 *years.*

10. To determine at what rate of interest an annuity of 50*l*. to continue 10 years, may be purchased for 400*l*.

Ans. 4.2775*l*. *per cent.* nearly.

11. Suppose an annuity of 175*l.* is to commence 9 years hence, and then continue 11 years: to find the present value, allowing 6 *per cent. per ann.* compound interest,

Ans. 816*l.* 18*s.* 9*d.* nearly.

12. A young man sinks 1000*l.* in purchasing an income of 100*l. per ann.* to continue till he is 60 years of age; now if he be 24 years old when the purchase money is paid, at what age will he begin to receive the annuity, allowing 5 *per cent. per ann.* compound interest? *Ans.* 32.127 *years,* nearly.

APPLICATION OF ALGEBRA TO GEOMETRY. *With the Solutions by* GEOMETRICAL CONSTRUCTION,

232. IN the preceding Articles we have considered Algebra as independent of Geometry, and demonstrated its operations from its own principles. We shall now explain the use of Algebra in resolving Geometrical Problems which depend on the properties of right lines and the circle. The student should therefore be master of the geometry in the first volume before he enters on this part.

Though the algebraic method is concise, and admirably adapted to the discovery of general properties and theorems, yet constructions purely geometrical claim the preference in point of elegance and perspicuity: this however, is to be understood of *plane problems* only, or such that may be resolved by a simple, or a quadratic equation. When the equation rises to a *cubic,* the problem is called a *solid* one, and the *actual* description of those lines by which the construction is effected in *that case,* becomes a matter of some difficulty.

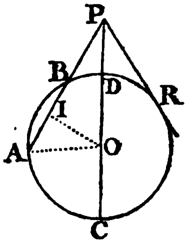
Different problems will require different methods of solution; and consequently it would not be easy to frame precepts for general reference. The student must therefore acquire his knowledge and dexterity in this branch from examples, and his own practice.

233. We shall begin with

The construction of the three forms of adfected Quadratic Equations:

$$\text{viz. } x^2 + ax = bc.$$
$$x^2 - ax = bc.$$
$$ax - x^2 = bc.$$

Construction of the first and second forms. With a radius equal to $\frac{1}{2}a$ let a circle be described, its centre being O. In this circle draw a chord AB $= b - c$ (b being supposed greater than c) and produce AB till BP $= c$; and from P draw PC through the center O. Then will $x =$ DP in the *first form*; and $x =$ CP in the *second*.



For AB $= b - c$, and BP $= c$, therefore AP $= b - c + c = b$. And since the radius of the circle is $= \frac{1}{2}a$, the diameter DC $= a$.

$$\text{Now (DP + DC)} \times \text{DP} = \text{AP} \times \text{BP}, \ (\textit{Geom. } 98)$$

that is $(x + a) \times x = b \times c$, or $x^2 + ax = bc$, the first form.

$$\text{And in the second form (PC} - \text{DC)} \times \text{PC} = \text{AP} \times \text{BP},$$

$$\text{or } (x - a) \times x = b \times c, \text{ that is } x^2 - ax = bc.$$

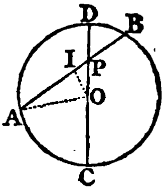
If the rectangle $bc = n^2$, or n is taken a geometrical mean between b and c, then the distance of the point P from the circle is found by drawing a tangent (RP) $= n$ from any point (R) in the circumference: for AP $\times$ BP $=$ PR^2 (*Geom.* 99) $= bc = n^2$. This latter method of construction (by means of the tangent) may be adopted when $b - c$ is greater than a the diameter of the circle;

Method of calculation. Draw the radius OA, and on AP let fall the perpendicular OI which will bisect the chord AB (*Geom.* 65).

Then $OI^2 = OA^2 - AI^2$ (*Geom.* 83); and $OP^2 = OI^2 + IP^2$, or $OP^2 = AO^2 - AI^2 + IP^2$; therefore $OP = \sqrt{(AO^2 - AI^2 + IP^2)}$, and consequently DP or $x = \sqrt{(AO^2 - AI^2 + IP^2)} - OD$ in the first form: and PC or $x = \sqrt{(AO^2 - AI^2 + IP^2)} + OC$ in the second.

Suppose $x^2 + 9x = 8 \times 4\frac{1}{2}$ (the first form).

Then $AO = \frac{1}{2}a = 4\frac{1}{2}$, $AI = \frac{b-c}{2} = 1\frac{3}{4}$, $IP = 6\frac{1}{4}$, OD or $OC = 4\frac{1}{2}$: and $x = \sqrt{(20\frac{1}{4} - 3\frac{1}{16} + 39\frac{1}{16})} - 4\frac{1}{2} = 7\frac{1}{2} - 4\frac{1}{2} = 3 = DP$; and $7\frac{1}{2} + 4\frac{1}{2} = 12 = PC$, the values of x in the two first forms.



Construction of the third form. Let a circle whose diameter is a be described as in the preceding forms; and take the chord $AB = b + c$; make $AP = b$, and $PB = c$, and through P draw the diameter DC. Then PD, and PC will be the two roots or values of x.

For $(DC - PD) \times PD = AP \times PB$, (*Geom.* 97)
that is $(a - x) \times x = b \times c$, or $ax - x^2 = bc$, (x being = PD).
And $(DC - PC) \times PC = AP \times PB$,
that is $(a - x) \times x$, or $ax - x^2 = bc$, (x being denoted by PC).

When $b + c$ is greater than the diameter a, take $n^2 = bc$, or let $ax - x^2 = n^2$, then AP and PB will be equal, and $AB = 2n$.

Method of calculation. From the centre O let fall the perpendicular OI upon AB; and join OA.

Then $OI^2 = AO^2 - AI^2$ (*Geom.* 83) $= \frac{1}{4}a^2 - \left(\frac{b+c}{2}\right)^2$:

And $IP = AP - AI = b - \frac{b+c}{2} = \frac{b-c}{2}$;

therefore $OP^2 = OI^2 + IP^2 = \frac{1}{4}a^2 - \left(\frac{b+c}{2}\right)^2 + \left(\frac{b-c}{2}\right)^2 = \frac{1}{4}a^2 - bc$;

and $OP = \sqrt{(\frac{1}{4}a^2 - bc)}$; whence PD, and PC the two values of x are $OD - OP$, and $OC + OP$, or $\frac{1}{2}a - \sqrt{(\frac{1}{4}a^2 - bc)}$; and $\frac{1}{2}a + \sqrt{(\frac{1}{4}a^2 - bc)}$.

Let $14x - x^2 = 8 \times 3$ be the equation. Then $a = 14 = DC$, $b = 8 = AP$, $c = 3 = PB$; and PD and PC will be 2, and 12 the values of x.

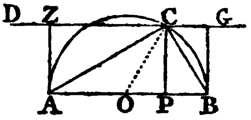
The same conclusions result from completing the square: thus, taking the third form $ax - x^2 = bc$, or $x^2 - ax = -bc$, whence $x^2 - ax + \frac{1}{4}a^2 = \frac{1}{4}a^2 - bc$, and $x = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - bc)}$.

From the preceding constructions it appears, that when a geometrical problem can be solved by an equation not exceeding a quadratic, it also admits of a construction by means of right lines and the circle.

234. *The area* (a) *of a right angled triangle, and the hypotenuse* (h) *being given, to find the other two sides.*

Suppose AB is the hypotenuse, and CP the perpendicular let fall from the right angle ACB upon the hypotenuse.

Since AB (or h) $\times$ CP $= 2a$ (*Mensur.* 257) we get $\frac{2a}{h} = CP$ the perpendicular.



Now let one of the segments AP or PB be denoted by x, then the other will be $h - x$. And because CP is a mean proportional between the segments AP, PB, (*Geom.* 164) we have $(h - x)x = \frac{4a^2}{h^2}$, or $x^2 - hx = -\frac{4a^2}{h^2}$; which equation gives $x = \frac{1}{2}h \pm \sqrt{\left(\frac{1}{4}h^2 - \frac{4a^2}{h^2}\right)}$, the two segments AP and PB. Whence the sides AC, BC will be found by *Geom.* 83.

Suppose the hypotenuse $AB = 13 = h$, and the area $= 39 = a$: then $\frac{1}{2}h \pm \sqrt{\left(\frac{1}{4}h^2 - \frac{4a^2}{h^2}\right)} = 6\frac{1}{2} \pm \sqrt{(42\frac{1}{4} - 36)} = 9$ and 4 the two segments. Now CP being $= \frac{2a}{h} = 6$, we have $AC = \sqrt{(81 + 36)}$, and $BC = \sqrt{(16 + 36)}$.

If the perpendicular $\frac{2a}{h} = \frac{1}{2}h$, the expression $\sqrt{\left(\frac{1}{4}h^2 - \frac{4a^2}{h^2}\right)}$ is $= 0$, and $x = \frac{1}{2}h$, or the point P coincides with the center of the circle: therefore should the given area exceed $\frac{1}{4}h^2$ the problem is impossible.

Otherwise thus:

Suppose O is the middle of AB; and put $x =$ OP, half the difference of the segments AP, PB; then $\frac{1}{2}h + x$ and $\frac{1}{2}h - x$ will denote the two segments; whence (*Geom.* 164), $(\frac{1}{2}h + x) \times (\frac{1}{2}h - x) = \frac{4a^2}{h^2}$ ($=$ CP²), which gives $x^2 = \frac{1}{4}h^2 - \frac{4a^2}{h^2}$, and $x = \sqrt{\left(\frac{1}{4}h^2 - \frac{4a^2}{h^2}\right)} =$ OP, which added to, and subtracted from $\frac{1}{2}h$, give the two segments AP and PB, as before.

Geometrically.

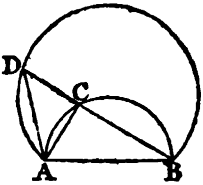
On the given hypotenuse AB describe a semicircle; also on the same line AB let a rectangle ZB be made equal to twice the area of the triangle (*Geom.* 179): from C draw CA, CB; and ACB is the triangle. For the triangle ACB is $= \frac{1}{2}$ the parallelogram ZB (*Geom.* 82. *corol.* 2), and the angle ACB a right one (*Geom.* 72.)

Method of calculation. Draw the radius OC; then the perpendicular CP being found as before; we have $\sqrt{(\text{OC}^2 - \text{CP}^2)} =$ OP (*Geom.* 83), or half the difference of the segments AP and PB: which is the same expression as that found by the last of the preceding methods.

In this construction, the vertex (c) of the triangle is determined by the intersection of two *loci*. Thus the circular arc ACB is called the *locus* or *place* of the angle ACB, because two lines drawn from A and B to meet any where in the arc ACB will form the same angle (*Geom.* 70). Therefore when the base, and vertical angle of a triangle are given, the *locus* of the vertex is the arc of a circle. But when the base and area are given, the *locus* of the vertex is a right line parallel to the base. Thus all triangles standing on the base AB, and having equal areas, will have their vertices in the indefinite line DG: *Geom.* 82. *corol.* 2.

235. *In a right-angled triangle* ACB, *let there be given the hypotenuse* AB, *and the sum of the other sides* AC + BC; *to determine those sides.*

Suppose AB $= h$, AC + BC $= s$, and BC $= x$.



Then $s - x =$ AC, and $(s-x)^2 + x^2 = h^2$ (*Geom.* 83) or $s^2 - 2sx + 2x^2 = h^2$; which equation solved gives $x = \frac{1}{2}s \pm \sqrt{\left(\frac{2h^2 - s^2}{4}\right)}$; and these two values of x are BC and AC.

If s^2 is greater than $2h^2$, the problem is impossible.

Suppose AB $= 15 = h$, and AC + BC $= 21 = s$:
Then $\frac{1}{2}s \pm \sqrt{\left(\frac{2h^2 - s^2}{4}\right)} = 10\frac{1}{2} \pm \sqrt{\frac{9}{4}} = 12$ and 9, the required sides.

Geometrically.

On the hypotenuse AB describe a semicircle, and also a segment of a circle to contain half a right angle; make the chord BD = the sum of the sides; then from the point of intersection C, draw CA; and ACB is the triangle.

For ACB is a right angle (*Geom.* 72), and therefore ACD is also a right angle; and since ADC is = half a right angle, the angle DAC must be half a right angle, and consequently AC = CD; therefore AC + CB = BD the sum of the sides by construction.

Method of calculation. By trigonometry, as AB : *sine* ADB :: BD : *sine* DAB; whence the angle DBA becomes known; and then the sides BC and AC are readily calculated from the given side AB.

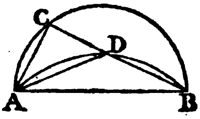
236. *Let the hypotenuse* AB, *and the difference of the sides* AC *and* BC, *be given; to find* AC *and* BC.

Put $h =$ the hypotenuse, $d =$ the difference of the required sides, and $x =$ the less side. Then $x + d$ will denote the greater.

And $(x + d)^2 + x^2 = h^2$ (*Geom.* 83), or $2x^2 + 2dx + d^2 = h^2$; this equation gives $x = \sqrt{(\frac{1}{2}h^2 - \frac{1}{4}d^2)} - \frac{1}{2}d$.

Suppose $h = 15$, and $d = 3$; then $\sqrt{(\frac{1}{2}h^2 - \frac{1}{4}d^2)} - \frac{1}{2}d = 10\frac{1}{2} - 1\frac{1}{2} = 9$ the least side; and $9 + 3 = 12$ the greater.

Geometrically.



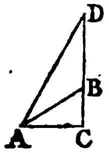
On the hypotenuse describe a semicircle, and also a segment to contain $1\frac{1}{2}$ right angles; and make the chord BD = the given difference of the sides: produce BD to C, and join CA; then ACB is the triangle.

For ACB being a right angle, and the angle ADC = ½ a right one, the angle DAC must be ½ a right one, and therefore AC = DC, and consequently DB is the difference of AC and CB.

Method of calculation. By trigonometry, as AB : *sine* ADB :: BD : *sine* DAB; this being determined, all the other angles in the figure become known.

237. *Having the base* (AC) *of a right angled triangle, and the sum of the hypotenuse and perpendicular* (AB + BC), *to find each of the latter sides.*

Let the base AC $= b$, AB + BC $= s$, and CB $= x$: Then the hypotenuse AB $= s - x$. And $s^2 - 2sx + x^2 = b^2 + x^2$ (*Geom.* 83); whence $s^2 - 2sx = b^2$, and $x = \frac{s^2 - b^2}{2s}$ the perp. CB: and $s - \frac{s^2 - b^2}{2s} = \frac{s^2 + b^2}{2s}$ the hypotenuse AB.



If AC $= 8 = b$, and AB + CB $= 16 = s$; then $\frac{s^2-b^2}{2s} = 6 =$ CB; and $\frac{s^2+b^2}{2s} = 10 =$ AB.

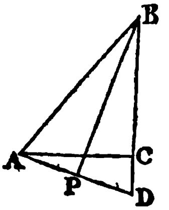
Geometrically.

At C the extremity of the given base, erect CD perpendicular to AC, and equal to the sum of the other sides; join AD, and make the angle BAD = BDA: Then ACB is the triangle. For the angle BAD being = the angle BDA, the opposite sides BD, BA must also be equal (*Geom.* 46. *cor.* 2) and therefore AB + BC = CD.

Method of calculation. By trigonom. DC : CA :: *radius* : *tang.* angle ADC; double this is the angle ABC (*Geom.* 41); therefore in the triangle ACB, the base AC, and all the angles will be given.

238. *Let the base* AC, *and difference of the hypotenuse* AB *and perpendicular* CB, *be given; to determine those sides separately.*

Suppose $b =$ the base, $d =$ the difference AB − BC, and $x =$ BC. Then $x + d =$ AB the hypotenuse.



And $(x+d)^2 = b^2 + x^2$, or $x^2 + 2dx + d^2 = b^2 + x^2$ (*Geom.* 83); whence $2dx = b^2 - d^2$, and $x = \frac{b^2-d^2}{2d} =$ CB: and therefore $\frac{b^2-d^2}{2d} + d = \frac{b^2+d^2}{2d} =$ AB the hypotenuse.

Geometrically.

Make CD perpendicular to the given base AC, and equal to the difference AB − BC; join AD which bisect in P, and draw PB perpendicular to AD, meeting DC produced in B; draw AB: and ACB is the triangle. For the angles at P being right ones, and PA = PD, therefore the triangle ABD is isosceles, and

AB = BD; therefore CD is the difference of CB and AB, the given difference by construction.

Method of calculation. By trigonom. AC : CD :: *radius* : *tang.* angle DAC; and its complement is the angle ADC = DAB, whence all the angles in the triangle ACB become known.

239. *To divide a given line* AB *into mean and extreme proportion.*

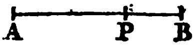
Let AB $= a$, and $x =$ AD the greater part, then $a - x =$ DB the less.

$$\text{And AB : AD :: AD : DB}$$
$$\text{or } a : x :: x : a - x, \text{ whence } x^2 = a^2 - ax,$$

and $x^2 + ax = a^2$; which gives $x = \left(\frac{5a^2}{4}\right)^{\frac{1}{2}} - \frac{1}{2}a = \text{AD}$.

For the *geometrical construction*, see *Geom.* 157.

* 239. *To divide a given line* AB *into two parts* AP *and* PB *so that their rectangle* AP × PB *shall be of a given magnitude* (m^2).



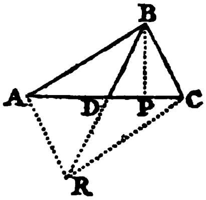
Let a denote the given line, and x one of the parts; then the other will be $a - x$; and $(a - x)\,x = m^2$, whence $x^2 - ax = -m^2$, which gives $x = \frac{1}{2}a \pm \sqrt{(\frac{1}{4}a^2 - m^2)}$; and these two roots or values of x are the parts AP and PB.

If m is greater than $\frac{1}{2}a$, the problem is evidently impossible.

The construction and method of calculation will be exactly the same as that in Art. 234, by taking PC $= m$.

240. THEOREM. *If a line* BD *be drawn from the vertex of any triangle* ABC *to bisect the base: Then* $2\text{BD}^2 + 2\text{AD}^2 = \text{AB}^2 + \text{BC}^2$.

Let BP be perpendicular to AC; and put BD $= b$, DB $= d$, and $h =$ AD or DC: then $h + d =$ AP, and $h - d =$ PC:



$$\begin{aligned} \text{And } BD^2 - DP^2 &= BP^2 \\ AP^2 + BP^2 &= AB^2 \quad (\textit{Geom. } 83.) \\ CP^2 + BP^2 &= BC^2 \end{aligned}$$

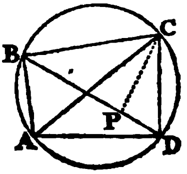
That is $b^2 - d^2 = DP^2$,

$$\begin{aligned} (h+d)^2 + b^2 - d^2 \text{ or } h^2 + 2hd + b^2 &= AB^2 \\ (h-d)^2 + b^2 - d^2 \text{ or } h^2 - 2hd + b^2 &= BC^2 \\ \text{whence by addition} \quad 2h^2 \quad + 2b^2 &= AB^2 + BC^2, \\ \text{or} \quad 2AD^2 + 2BD^2 &= AB^2 + BC^2. \end{aligned}$$

Corol. If AR, CR be parallel to BC, AB, respectively; then BDR and AC will be the diagonals of the parallelogram ABCR, and DR $=$ DB (*Geom.* 81. *corol.* 2), and therefore $2DC^2 + 2RD^2 = CR^2 + RA^2$; but DC $=$ AD, and BD $=$ RD; consequently $4DC^2 + 4RD^2 = CR^2 + RA^2 + AB^2 + BC^2$: but 4 times the square on half a line is equal to the square on the whole line; therefore $4DC^2 = AC^2$, and $4RD^2 = RB^2$; whence we have $AC^2 + RB^2 = CR^2 + RA^2 + AB^2 + BC^2$; that is, *the sum of the squares of the two diagonals of a parallelogram, is equal to the squares on the four sides taken together.*

241. THEOREM. *The rectangle under the two diagonals of any quadrilateral inscribed in a circle, is equal to the sum of the two rectangles of the opposite sides:*

That is, AC × BD = AB × CD + AD × BC.



Suppose CP is drawn to make the angle PCD = BCA:

Then because the angle PDC = BAC, (*Geom.* 70), the triangles CPD, CBA are equiangular; whence AC : AB : : DC : DP (*Geom.* 94. *corol.* 1), therefore AC × DP = AB × DC.

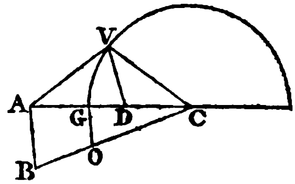
Again; because the angle $CPD = CBA$, and $CPD + CPB$ make two right angles, and $CBA + CDA$ also make two right angles, the angle $CPB = CDA$, and the angle $PBC = CAD$ (*Geom.* 70), therefore the triangles CPB, CDA are equiangular; consequently $AC : AD :: BC : BP$, whence $AC \times BP = AD \times BC$; now by adding this equation and the former together, we have

$$AC \times BP + AC \times DP = AD \times BC + AB \times DC,$$

that is $(BP + DP) \times AC$, or $BD \times AC = AD \times BC + AB \times DC$.

242. THEOREM. *Let* ABC *be any triangle, and suppose* $AG = AB$, *and* GO *parallel to* AB; *then if* GD *be taken* $= GO$, CG *will be a mean proportional between* AC and CD.

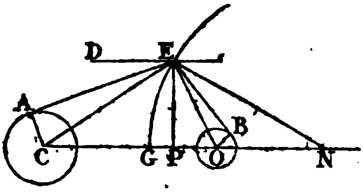
By similar triangles, $CA : AB\ (AG) :: CG : GO\ (GD)$,
and by division, $CA : CA - AG :: CG : CG - GD$,
or $CA : CG :: CG : CD$.



From this Theorem the *locus* of the vertex of a triangle may be determined when the base AD and the *ratio* of the sides AV, DV are given. Thus, if a circle be described about the center C with the radius CG, then any two lines drawn from A and D to meet in the circumference will have the ratio of AG to GD.

For $CA : CG\ (CV) :: CG\ (CV) : CD$; that is, the sides of the triangles CVA, CVD about the common angle at C, are proportional, and therefore the triangles are similar (*Geom.* 94. *cor.* 1), consequently the other sides are proportional, *viz.* $AV : DV :: CA : CV\ (CG) :: AB\ (AG) : GO\ (GD.)$ Hence, to describe the circle which is the required *locus*, divide the base in the given proportion of the sides, and find the center (C) as above.

243. If a 9lb. iron shot, and another 48lb. are on the ground at 20 yards distance from each other; where must I stand in a line between them so that each may appear of the same magnitude; the height of the eye being 5½ feet?



Suppose O to be the center of the less shot, C the center of the greater, and E the place of the eye; also let EA, EB be tangents at A and B; then CAE, OBE will be right angles.

Now it is evident that when the two shot appear of the same magnitude, the diameters of the circles which bound the visible surfaces will be seen under equal angles, or, which amounts to the same thing, the angles CEA, OEB will be equal; therefore the triangles CAE, OBE will be similar, and consequently CE, OE will have the same ratio as the radii CA, OB, or as 3½ *inches* to 2 *inches*, which are the radii of a 48*lb.* shot, and a 9*lb.* shot, nearly.

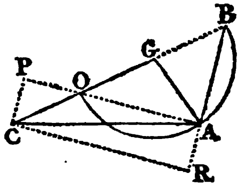
Construction. Divide the distance CO = 20 yards into two parts CG, GO having the ratio of 3½ to 2, and describe the *locus* of the vertex of the triangle CEO, as directed in the preceding article: then if DE be drawn parallel to, and at the distance of 5½ *feet* from CO, the point E where it intersects the circle, will be the place of the eye.

Calculation. CG = 38.182, and GO = 21.818 *feet*, nearly.

And 38.182 — 21.818 : 38.182 :: 38.182 : 89.089 = CN, from this take CG, and there remains 50.907 = GN = NE the radius; then the perpendicular PE being = 5½ *feet*, we get PN = 50.6, which taken from CN gives CP = 38.489 *feet*, the distance from the greatest ball where a person must stand to see them both under the same angle.

N. B. The point P is between 2 and 3 *inches* from the ground, and consequently PE is taken that quantity too great in the computation; the conclusion however, is not materially affected on that account.

244. *Having the sides of a plane triangle, to find the angles.*



Let CGA be the triangle. About G as a center with the radius GA describe a semicircle; produce CG to B; draw BAR, and AOP; make CP perpendicular to AP, and CR perpendicular to BR; then the angle OAB in a semicircle is a right one, consequently PA, CR are parallel, and PC = AR; also GO = GA = GB; and CB is the sum, and OC the difference of the sides GC, GA; and because the triangle AGB is isoceles, the angle GBA is = $\frac{1}{2}$ the angle CGA.

Now if AB be the base of the triangle ACB, and CR the perpendicular on the base (produced), we have (*Trigonom.* art. 228) $AB : BC + AC :: BC - AC : \frac{BC^2 - AC^2}{AB}$, and $\frac{BC^2 - AC^2}{2AB} - \frac{AB}{2} = AR = CP$.

Again, suppose AO is the base of the triangle ACO, and CP the perpendicular on the base (produced),

then $AO : AC + CO :: AC - CO : \frac{AC^2 - CO^2}{AO}$, and $\frac{AC^2 - CO^2}{2AO} - \frac{AO}{2} = OP$.

But the triangles CPO, BAO are similar,

that is $CP : AB :: OP : AO$;

$$\text{or } \frac{BC^2 - AC^2}{2AB} - \frac{AB}{2} : AB :: \frac{AC^2 - CO^2}{2AO} - \frac{AO}{2} : AO;$$

$$\text{or } \frac{BC^2 - AC^2}{AB} - AB : AB :: \frac{AC^2 - CO^2}{AO} - AO : AO;$$

$$\text{and by composition, } \frac{BC^2 - AC^2}{AB} : AB :: \frac{AC^2 - CO^2}{AO} : AO;$$

whence $BC^2 - AC^2 : AB^2 :: AC^2 - CO^2 : AO^2$, (by equimultiples), or $BC^2 - AC^2 : AC^2 - CO^2 :: AB^2 : AO^2$, that is $(BC+AC)(BC-AC) : (AC+CO)(AC-CO) :: AB^2 : AO^2$:

But if AB be made *radius*, AO will be the *tangent* of the angle OBA or half the angle CGA of the proposed triangle CGA.

Therefore let s and d respectively denote the *sum* (BC) and *difference* (CO), of the sides including the required angle (CGA), and b the other side (AC):

Then $(s+b)(s-b) : (b+d)(b-d) :: rad.^2 : \frac{(b+d)(b-d)}{(s+b)(s-b)} \times rad.^2 =$ the square of the tangent of half the angle CGA.

Example. Let CA = 462, CG = 384, AG = 169; and the radius = 1.

$$\begin{array}{rl@{\qquad\qquad}rl}
384 & & 384 & \\
169 & & 169 & \\
\hline
553 & = s & 215 & = d \\
462 & = b & 462 & = b \\
\hline
1015 & = s+b & 677 & = b+d \\
91 & = s-b & 247 & = b-d \\
\hline
\end{array}$$

$\frac{(b+d)(b-d)}{(s+b)(s-b)} \times rad.^2 = \frac{677 \times 247}{1015 \times 91} = 1.8104$ nearly, and its square root $= 1.3455$ the *natural tangent* of 53° 23′.

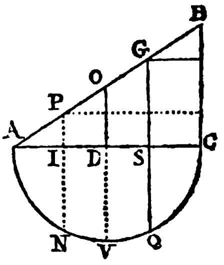
But the operation by logarithms is very short:

1015	*ar. co. log.*	6.993534
91	*ar. co. log.*	8.040959
677	.... *log.*	2.830589
247	.... *log.*	2.392697
$rad.^2$	.... *log*	20.000000
	2)	20.257779
53° 23′	*tang. log.*	10.128889

Therefore twice 53° 23′ is 106° 46′ the required angle CGA.

This rule is preferable to that in vol. 1, art. 229; and should be used when the required angle is very obtuse.

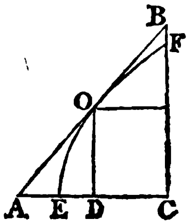
245. *In a right angled triangle* ACB *to inscribe a rectangle* GC *of a given magnitude* (m^2).



Let OD, parallel to BC, bisect AC the base in D. Then by similar triangles, $AD : DO :: AD + DS : \frac{DO}{AD} \times (AD + DS)$ $= SG$, which drawn into SC or $AD - DS$ is the rectangle GC, that is $\frac{DO}{AD} \times (AD + DS)\ (AB - DS) = m^2$, or $DO : AD :: m^2 : (AD + DS)\ (AD - DS)$. Hence the following *construction:*

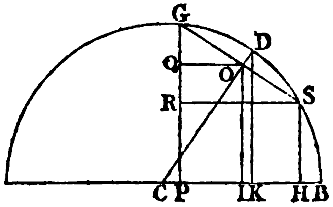
On AC describe a semicircle, in which, at right angles to AC, apply SQ and IN so that $DO : AD :: m^2 : SQ^2$ or IN^2 (Geom. art. 186); then if SG, IP are perpendicular to AC, either of the rectangles GC, PC is that required. For $(AD + DS)\ (AD - DS) = SQ^2$, by Geom. 97, corol. 1. Therefore $\frac{DO}{AD} \times SQ^2 = m^2$.

Corol. 1. When $m^2 = AD \times DO$, then $DO : AD :: DO \times AD : AD^2$ (or DV^2), and the points I, S, coincide in D, and DO, DC are the sides of the rectangle, which is a *maximum*, because DV is the greatest possible. And in that case, the problem admits of but one answer.



Corol. 2. Hence we conclude, that if EOF be any curve concave to its axis EC, the greatest inscribed rectangle OC will be when OD bisects the base AC, or the subtangent $DA = \frac{1}{2}$ the base, and the tangent AB is bisected at the point of contact O.

246. *The sines and cosines of two arcs* (BD, DG) *being given, to find the sine and cosine of the sum* (BG), *and difference* (BS) *of those arcs.* (See Trigonom. art. 210).



Draw the sines SH, DK, GP, and the radius CD; also join SG, and let SR and OQ be parallel to the radius CB, and OI perpendicular to CB. Then DK, KC; and GO, OC, are the *sines*, and *cosines* of the arcs BD, DG.

By similar triangles:

$CD : CO :: DK : OI$, whence $\frac{CO.DK}{CD} = OI = QP$.

$CD : CK :: GO : GQ$, and $\frac{CK.GO}{CD} = GQ$: therefore

$\frac{CO.DK}{CD} + \frac{CK.GO}{CD} = QP + GQ = PG$ *the sine of the sum*;

and $\frac{CO.DK}{CD} - \frac{CK.GO}{CD} = QP - QG\ (QR) = RP = SH$, *the sine of the diff.*

Again;

$CD : CK :: CO : CI$, whence $\frac{CK.CO}{CD} = CI$,

$CD : DK :: GO : OQ$, and $\frac{DK.GO}{CD} = OP = PI = IH$,

whence $\frac{CK.CO}{CD} - \frac{DK.GO}{CD} = CI - IH = CP$, *cosine of the sum*;

and

$\frac{CK.CO}{CD} + \frac{DK.GO}{CD} = CI + IH = CH$, *cosine of the difference.*

Hence, if the *sine* and *cosine* of an arc be denoted by S and C, and the *sine* and *cosine* of another arc by s and c, and the *radius* $= r$:

Then, the *sine* of their *sum* $= \frac{Sc + sC}{r}$:

cosine $= \frac{Cc - Ss}{r}$.

sine of their *diff.* $= \frac{Sc - sC}{r}$;

cosine $= \frac{Cc + Ss}{r}$.

When the given arcs are equal, then $S = s$, and $C = c$, and the expressions become $\frac{2sc}{r}$, and $\frac{c^2 - s^2}{r}$, or (making the radius $= 1$) $2sc$ and $c^2 - s^2$ for the *sine* and *cosine* of *double* the arc whose *sine* and *cosine* are denoted by s and c.

Therefore, if A be the arc whose *sine* is s, and *cosine* c; then the *sine* and *cosine* of $A + A$ (or $2A$) will be $2sc$, and $c^2 - s^2$:

And by the same rule, the *sine* of $(A + A) + A$ (or $3A$) is $2sc \times c + s\,(c^2 - s^2)$, or $3s\,c^2 - s^3$:

And the *cosine* $(c^2 - s^2)\,c - 2sc \times s$, or $c^3 - 3sc$. That is; if any *sine*, or *cosine*, be multiplied by $2c$, and the next preceding one by $s^2 + c^2$, and the latter product subtracted from the former, the remainder is the next following *sine*, or *cosine*.

Hence we have

	sines.	*cosines.*
Arc A	s	c
$2A$	$2sc$	$c^2 - s^2$
$3A$	$3sc^2 - s^3$	$c^3 - 3s^2c$
$4A$	$4sc^3 - 4s^3c$	$c^4 - 6s^2c^2 + s^4$
$5A$	$5sc^4 - 10s^3c^2 + s^5$	$c^5 - 10s^2c^3 + 5s^4c$
$6A$	$6sc^5 - 20s^3c^3 + 6s^5c$	$c^6 - 15s^2c^4 + 15s^4c^2 - s^6$.
	&c.	&c.

Where the law of continuation is manifest, it being such, that all the terms of the *cosine* and *sine* of any multiple arc nA (n being a positive integer) taken in order, are the terms of the binomial $(c + s)^n$. Thus, for example, to find the *sine* and *cosine* of $5A$:

$(c + s)^5 = c^5 + sc^4 + 10s^2c^3 + 10s^3c^2 + 5s^4c + s^5$, where the first, third, fifth, &c. terms being set down alternately +, and —, is the *cosine*: and the second, fourth, sixth, &c. connected in like manner, give the *sine*.

247. *The tangents of two arcs being T and t; to find the tangent of the sum of those arcs, and also of their difference.*

If the corresponding *sines* and *cosines* are denoted as in the preceding problem; then by *Trigonom.*

cosine : *sine* :: *radius* : *tang.* (*Geom.* 204)

viz. $Cc - Ss : Sc + sC :: 1$ (*radius*) : $\dfrac{Sc + sC}{Cc - Ss}$, *tang.* of the sum.

$Cc + Ss : Sc - sC :: 1 : \dfrac{Sc - sC}{Cc + Ss}$, *tangent* of the difference.

Now suppose m and n to represent the secants of the given arcs, respectively;

Then (*Geom.* 211)

$m : T :: 1$ (*radius*) : $\dfrac{T}{m} = S$ the *sine*:

$m : 1 :: 1 : \dfrac{1}{m} = C$ the *cosine.*

And in like manner we have $\dfrac{t}{n} = s$, and $\dfrac{1}{n} = c$: these values being substituted in the foregoing expressions for the tangents,

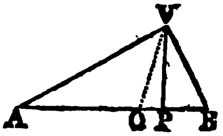
give $\dfrac{T + t}{1 - Tt}$ the *tangent* of the *sum.*

and $\dfrac{T - t}{1 + Tt}$ the *tangent* of the *difference.*

Remark. If an arc be greater than 90°, its tangent, cosine, &c. are to be used with the negative sign prefixed.

248. *Given the vertical angle* (AVB), *the perpendicular* (VP), *and the rectangle of the segments of the base made by the perpendicular,* (AP × PB); *to determine the triangle.*

Let VO bisect the verticle angle, and put $t =$ the *tangent* of AVO or OVB, and $x =$ the *tangent* of OVP; the *radius* being 1.



Then by the preceding problem $\frac{t+x}{1-tx}$ is the *tang.* of the angle AVP, and $\frac{t-x}{1+tx}$ that of the angle PVB.

Now if the rectangle of the segments be denoted by r, and the perpendicular VP by p, we shall have (by *trigonom.*)

$$1 : \frac{t+x}{1-tx} :: p : \frac{pt+px}{1-tx} = \text{PA}:$$

$$1 : \frac{t-x}{1+tx} :: p : \frac{pt-px}{1+tx} = \text{PB}:$$

Therefore $\frac{pt+px}{1-tx} \times \frac{pt-px}{1+tx} = \frac{p^2t^2-p^2x^2}{1-t^2x^2} = r;$

Whence, by reduction, $x = \sqrt{\frac{r-p^2t^2}{rt^2-p^2}}$ the *tang.* of OVP. This added to, and subtracted from $\frac{1}{2}$ the vertical angle, will give the angles AVP, and PVB; and thence the segments of the base, &c. may readily be found by trigonometry.

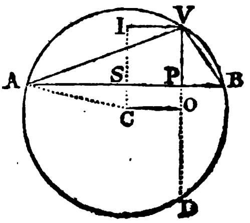
If p^2t^2 be greater than r, the problem is evidently impossible.

Example. Suppose the vertical angle = 95°, the perpendicular = 60, and the rectangle of the segments of the base = 4400:

Then $t = 1.0913085$ the natural *tang.* of $47\frac{1}{2}°$; and $\sqrt{\frac{r-p^2t^2}{rt^2-p^2}} = .26196$ the natural *tang.* of 14° 41′, nearly, the angle OVP; whence AVP = 62°. 11′, and PVB = 32° 49′. And the sides are found by common trigonometry.

Geometrically.

Suppose AVB is the triangle circumscribed by a circle whose center is C. Let the perpendicular VP be produced to D: then since AP × PB is given, and = VP × PD (*Geom.* 97), therefore PD is given, and consequently VD will

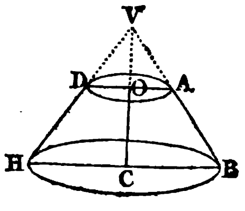


also be known. Join CA, and draw CO parallel to AB, and CS perpendicular to AB, then VD and AB are bisected in O and S, respectively (*Geom.* 65), and consequently CS = PO the difference of $\frac{1}{2}$ VD and the perpendicular PV. Also, the angle CAS is the difference of AVB and a right angle (*Geom.* 172). Hence the following *construction* is obvious:

Make the given rectangle AP × PB = m^2 (*Geom.* 184); and we get VP : m :: m : PD; then if a right angled triangle ASC be constructed having the angle SAC = the difference of the given vertical angle and a right one, and the opposite side SC = the difference of VP and $\frac{1}{2}$ VD, the side AS will be $\frac{1}{2}$ the base AB, and AC the radius of the circumscribing circle, which describe, and produce CS till SI = the given perpendicular; then draw IV parallel to the base AB; and the point V in the circle will be the vertex of the required triangle AVB.

Calculation. Taking the *data* as before, we have $\frac{AP \times PB}{VP} = \frac{4400}{60} = 73\frac{1}{3}$ = PD, whence PO or SC = $6\frac{2}{3}$; and the angle CAS = 5°, therefore (by *trigonom.*) as *radius* : *cotang.* 5° :: $6\frac{2}{3}$ (CS) : 76.2 = AS half the base. And, as *sin.* 5° : $6\frac{2}{3}$ :: *rad.* : 76.4914 = CA the radius of the circle, which also is the diagonal of the parallelogram CIVO; and VO being $66\frac{2}{3}$, we get CO = SP = 37.5 nearly, consequently the segment AP = 76.2 + 37.5 = 113.7, and PB = 76.2 − 37.5 = 38.7, nearly; whence AV and VB are readily found.

249. *To investigate the content of the frustum of a Cone or Pyramid.*



Let HDAB be the frustum, and suppose the cone or pyramid to be completed.

Put b^2 = the area of the base HB, t^2 = that of the top DA, h = the height CO, and x = OV, CV being the axis, Then $b^2 : t^2 :: (h+x)^2 : x^2$ (*Geom.* 132)

or $b : t :: h+x : x$, whence $x = \frac{ht}{b-t} =$ OV, therefore CV $= h + \frac{ht}{b-t} = \frac{hb}{b-t}$.

And $\frac{1}{3}b^2 \times \frac{hb}{b-t} = \frac{\frac{1}{3}hb^3}{b-t}$ the content of HVB, (Geom. 133. *cor.* 2)

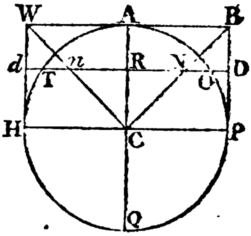
Also $\frac{1}{3}t^2 \times \frac{ht}{b-t} = \frac{\frac{1}{3}ht^3}{b-t}$ the content of DVA:

And the difference, or $\frac{\frac{1}{3}hb^3 - \frac{1}{3}ht^3}{b-t} = \frac{1}{3}h\left(\frac{b^3-t^3}{b-t}\right) = \frac{1}{3}h(b^2+bt+t^2)$,

viz. *The areas of the two ends and the mean proportional between them in one sum, multiplied by $\frac{1}{3}$ of the height*, is the content of the frustum HDAB. The same result as that in *art.* 283 mensuration.

250. *To find the content of a segment of a sphere.*

Let HP be the diameter of a sphere; TAO a segment; and suppose the hemisphere to be circumscribed by a cylinder HWBP. Put RA the height of the segment $= h$; TR or RO the radius of its base $= b$; and $m = .7854$.



Then $\frac{b^2}{h} =$ RQ (*Geom.* 97), and $\frac{b^2}{h} + h = \frac{b^2+h^2}{h} =$ AQ $=$ WB the diam. of the sphere, and $\frac{b^2+h^2}{2h} =$ the radius CA; whence $\frac{b^2+h^2}{2h} - h = \frac{b^2-h^2}{2h} =$ RC $=$ RN $=$ Rn, and $\frac{b^2-h^2}{h} =$ nN.

Now $\left(\frac{b^2+h^2}{h}\right)^2 \times m + \left(\frac{b^2-h^2}{h}\right)^2 \times m + \frac{b^2+h^2}{h} \times \frac{b^2-h^2}{h} \times m \times \frac{1}{3}h$ is the content of the conic frustum nWBN (*by the preceding article*):

Also $\left(\frac{b^2+h^2}{h}\right)^2 \times m \times h$ is the content of the cylinder dWBD:

And (*Geom.* 134. *cor.* 2) the difference of those expressions is the content of the segment TAO;

that is $\left(2\left(\frac{b^2+h^2}{h}\right)^2-\left(\frac{b^2-h^2}{h}\right)^2-\frac{b^2+h^2}{h}\times\frac{b^2-h^2}{h}\right)\times\frac{1}{3}mh$,

which, by reduction, becomes $\frac{6b^2+2h^2}{3}\times mh$, or, putting $B=2b=$TO, we have $(3B^2+4h^2)\times h\times\frac{1}{6}m$ the content of the segment.

In words, *To* 3 *times the square of the diameter of the base add* 4 *times the square of the height, then multiply the sum by the height, and that product by the decimal* .1309 (or $\frac{1}{6}$ of .7854), and the result is the content.

251. *A stone being let fall into a well, it was observed, that after being dropped, it was* 4 *seconds before the sound of the fall at the bottom reached the ear. Hence the depth of the well is required.*

Heavy bodies near the earth's surface descend by their own gravity $16\frac{1}{12}$ *feet* in the first second of time, $48\frac{3}{12}$ in the next, $80\frac{5}{12}$ in the third, and so on, constituting a series of distances in arithmetical progression, the first term being $16\frac{1}{12}$, and common difference $32\frac{2}{12}$. Now let $f=16\frac{1}{12}$, $d=32\frac{2}{12}$, $t=4$ *seconds*, and $x=$ the sum of all the terms in the progression or depth of the well; also put $s=1100$ *feet* the velocity of sound *per second:*

Then (138) we have $\sqrt{\frac{2x}{d}}$ for the number of terms or time in which the stone is falling to the bottom, ($2f=d$ in this case); and $\frac{x}{s}$ the time of the sound's ascent: therefore both times together must be equal to the whole time t;

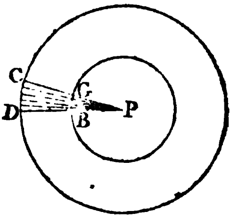
viz. $\sqrt{\frac{2x}{d}}+\frac{x}{s}=t$, whence $\sqrt{\frac{2s^2x}{d}}+x=st$.

Now put $r=\sqrt{\frac{2s^2}{d}}$, and $z^2=x$;

then $rz+z^2=st$; whence we have $z=\sqrt{(\frac{1}{4}r^2+st)}-\frac{1}{2}r$,

or $z^2=x=(\sqrt{(\frac{1}{4}r^2+st)}-\frac{1}{2}r)^2=231$ *feet*, nearly, by substituting the preceding numeral values of the different letters.

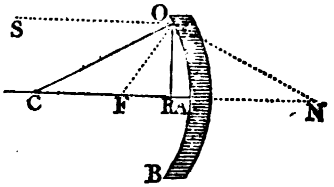
252. *To find the comparative intensity of light at different distances from a lucid point* P.



Let BPG and DPC be similar sectors of two circles described about P as a center.

Now the same light which issues from the point P, and is spread over the sector BPG, will also be diffused through the sector DPC. And it is evident the intensity of the light must grow less, as the space it occupies becomes greater: but the spaces BPG, DPC are as the squares of the radii PB and PD (Geom. 105. *corol.*); hence we conclude, that *the intensities are inversely as the squares of the distances*; that is, if the light at B and D are respectively denoted by N and n, then $PB^2 : PD^2 :: n : N$. And the same conclusion is evident in respect of *heat.* Thus if PD is double PB, then the heat at B will be 4 times that at D, supposing P is a point from which heat is emitted.

253. *Let* OAB *be a concave spherical speculum or mirror; to find the point* F *where a ray of light* SO *parallel to the axis* AD *meets the radius* CA *after being reflected from the incident point* O.



Let C be the centre of the spherical surface OAB; CO a radius, and OR perpendicular to the radius CA. Then because CO is perpendicular to the surface of the speculum at O, the angle of reflexion COF is equal to the angle of incidence COS, or the direct and reflected rays SO, OF make equal angles with the perpendicular CO. This is one of the known laws of optics.

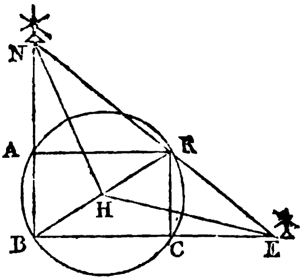
Because SO is parallel to CR, the angles SOC, RCO are equal; but the angle COF = COS, therefore the triangle CFO is isoceles, and FO = FC. Suppose ON = OC, then the triangle CON is isoceles, and RN = RC: moreover, as the angle at C is common to both the isoceles triangles, they must be equiangular or similar; hence CN (2CR) : CO :: CO : CF. Let the radius CO $= r$, CR $= b$, and $x =$ CF; then the proportion is, $2b : r :: r : \frac{r^2}{2b} = x =$ CF, whence AF $= r - \frac{r^2}{2b}$ $= \frac{2br - r^2}{2b}$ the distance from the speculum where the reflected ray OF meets the axis.

If the incident ray is indefinitely near the axis DA, then b may be taken $= r$, and x becomes $= \frac{1}{2}r$, or FC = FA.

Suppose the radius CO or $r = 6$ *feet* $= 72$ *inches*, and OR $= 20$ *inches*, then CR $= 69.166$ *inches* nearly, $= b$; whence CF $= 37.47$ *inches*. Hence it appears that when a spherical speculum or burning mirror is exposed to the rays of the sun, all the reflected rays are not collected into one point or *focus*, for the extreme rays, or those reflected from O and B, meet the axis at a greater distance from the centre C than those that are incident near the middle of the speculum at A. Thus in the present instance, the point F is 1.47 *inches* from the middle of the radius CA.

The *Geometrical construction* is obvious: For having drawn an incident ray SO (whether it be parallel to the axis CA, or not) join O, C, and make the angle of reflexion COF equal to the angle of incidence COS; then F is the point where the reflected ray meets the axis.

254. *In reconnoitring a country we observed two windmills, one bore* N. *the other* E. *we then proceeded 2 miles in a* NEbE *direction and came to a village that was at an equal distance from those objects; and when we had continued our route 2 miles further in the same direction, we found ourselves upon an height in a line directly between them. Hence the distance from one windmill to the other is required?*



Let N and E be the north and east windmills, B the place where they were first observed, BR the NEbE line, H the village, and R the height in the line NE. About H with the radius HB or HR (2 *miles*) describe a circle, and join RC, RA.

Then, since HE = HN, the points E, N, are equally distant from the circle, and therefore it follows from *Geom.* 98 or 99, that the rectangle (BC + CE) × CE is equal to the rectangle (BA + AN) × AN; whence we have

BC + CE : BA + AN :: AN : CE.

But the triangles ANR, CRE are similar,

whence AR (BC) : CE :: AN : RC (BA);

and by composition, BC + CE : BA + AN :: CE : BA;

therefore by equality (from the first proportion)

CE : BA :: AN : CE; consequently CE is a mean proportional between BA (or RC) and AN.

Also, by similar triangles, RC : CE :: AN : AR (BC); now the 2*d.* term CE being a mean proportional between the 1*st.* and 3*d.* terms RC and AN, the 4 terms RC, CE, AN, BC, are continued proportionals: Hence the question is reduced to that of finding 2 mean proportionals between two given lines (RC, BC), which is a *solid problem*, and consequently will not admit of a geometrical construction by means of right lines and the circle only (232).

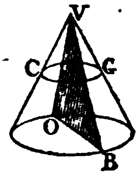
If $d = BR = 4$, $x =$ CE the 2d. term, and s and c are put to denote the *sine* and *cosine* of 33°.45′ the angle RBC (the *radius* being 1); then by *trigonom.* $sd =$ RC, and $cd =$ BC; and the 4 proportionals are sd, x, $\frac{x^2}{sd}$, cd, whence $x \times \frac{x^2}{sd} = scd^2$, which gives $x = \left(cs^2d^3\right)^{\frac{1}{3}} = 2.54196$ *miles*, nearly, = CE; whence by trigonometry, NE is found = 7.7946 *miles*.

CONIC SECTIONS.

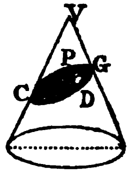
DEFINITIONS.

255. The figures denominated Conic Sections are made by a plane cutting a cone.

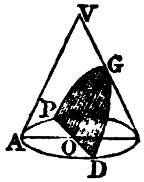
256. If the plane pass through the vertex of the cone, the section will evidently be a triangle. And if it be parallel to the base of the cone, the section is a circle (*Geom.* 121. *corol.*). Thus the section OVB is a triangle; and the section CG a circle. These however, are not called conic sections.



257. When the plane is inclined to the base of the cone, and cuts both sides, the section is an *ellipse*. Thus CG is an ellipse.

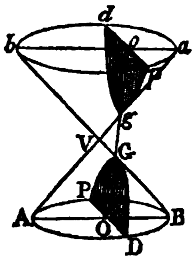


258. If the intersecting plane is parallel to the side, the section is called a *parabola*. Thus if the plane PGD is parallel to VA, the section PGD is a parabola.



259. But if the inclination of the plane be such that it cuts the *opposite cone* when produced, the section is an *hyperbola*.

Thus if V be the vertex common to both cones VAB, V*ab*, and the plane PDG when produced cuts the cone V*ab*, then PDG, *pgd*, are opposite hyperbolas. This section, the parabola, and ellipse, are exclusively called *Conic Sections*.



260. The *vertices* of any section are the points where the intersecting plane meets the opposite sides of the cone.

Thus C and G are the vertices of the ellipse, and G, *g*, those of the opposite hyperbolas. The parabola has only one vertex G.

261. The *axis* or *transverse diameter* is the line connecting the vertices.

Thus CG is the axis of the ellipse; and G*g* that of the hyperbola. But the axis of the parabola is infinite in length; GO being a part of that axis.

262. The *center* of any section is the middle of the axis; consequently that of the parabola is at an infinite distance from the vertex.

263. A *diameter* is any right line passing through the centre and terminated by the curve.

264. An *ordinate* to a diameter is any right line terminated by the curve, and bisected by that diameter.

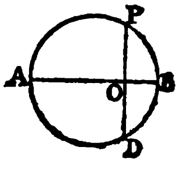
Thus PD is an ordinate to the axis CG in the ellipse, to G*g* in the hyperbola, and to GO in the parabola. The semi-ordinates OP, or OD, or O*d*, are also called ordinates.

Hence the ordinates to the axis are at right angles to it.

265. An *absciss* or *abscissa* is a part of any diameter contained between its extremities and an ordinate to it.

Thus OG, O*g*, are abscissas.

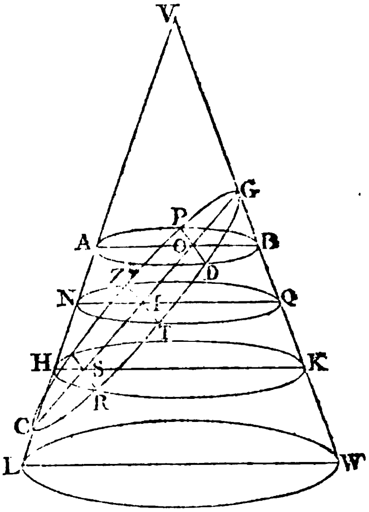
266. BEFORE we proceed to the properties of the sections, it may be necessary to explain what is understood by the *equation of a curve*.



Let AB the diameter of a circle bisect the chord DP; then OD = OP, and AO × OB = OD^2 (*Geom.* 97. corol. 1), or (AB − OB) × OB = OD^2; therefore if the diameter AB = d, AO or OB = x, and OD = OP = y;

then $(d-x)\times x=y^2$, or $dx-x^2=y^2$ is the *equation of the circle*, and defines the nature of that curve by expressing the relation of the *absciss* x to the corresponding *ordinate* y at any point (O) in the diameter.

To find the Equations of the three Conic Sections.



267. Let the plane CPGDR cut the cone VLW; and suppose GO = CS; also let PD and SR be at right angles to GC, and the points P and D at equal distances from the vertex V: then if AB and HK are the diameters of the circular sections through O and S, PD and SR will be in the planes of those circles, and perpendicular to the diameters AB, HK, respectively.

Then the triangles SGK, OGB; OAC, SHC, being respectively similar, we have

$$SK : OB :: SG\ (OC) : OG\ (SC) :: OA : SH,$$

$$\text{whence } SK \,.\, SH = OB \,.\, OA.$$

And since OD and SR are in the planes of the circular sections whose diameters are AB, and HK, and also perpendicular to those diameters, therefore (Geom. 97. *corol.* 1) $SK \,.\, SH = SR^2 = OB \,.\, OA = OD^2$, consequently SR and OD are equal; that is, whatever be the curve CPGDR, the ordinates (SR, OD) to the axis CG, are equal at equal distances from the vertices C and G.

Now suppose NQ is the diameter of another circular section of the cone; then IT (in the plane of that circle, and also in that of the oblique section CPGTR) will be perpendicular to CG and NQ, whence, by similar triangles,

SC : SH :: IC : IN,
SG : SK :: IG : IQ,

therefore (96) SC.SG : SH.SK (SR^2) :: IC.IG : IN.IQ (IT^2);
or SR^2 : SC.SG :: IT^2 : IC.IG :

That is, *As the square of any ordinate* (SR^2),
Is to the rectangle of the corresponding abscissas (SC . SG),
So is the square of any other ordinate (IT^2),
To the rectangle of its corresponding abscissas (IC . IG).

If I be the center of the transverse axis CG, then TZ (at right angles to CG) is called the *conjugate* axis.

Let CG $= t$, TZ $= c$, any abscissa (OG or OC) $= x$, and the corresponding ordinate (OD) $= y$. Then the preceding analogy becomes

$$\tfrac{1}{2}t \times \tfrac{1}{2}t : \tfrac{1}{2}c \times \tfrac{1}{2}c :: (t-x)x : y^2,$$

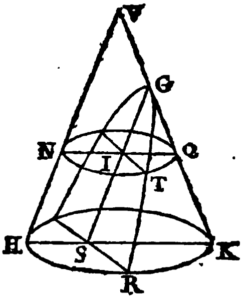
$$\text{or } t^2 : c^2 :: tx - x^2 : y^2;$$

that is $\frac{c^2}{t^2}(tx-x^2) = y^2$. Which is the *equation of an ellipse*, or the *oval* CZPGDTR.

268. When the intersecting plane is parallel to the side of the cone, (PD being at right angles to CG, and its extremities P and D equally distant from the vertex of the cone, as before) then C will be at an infinite distance, and the axis infinite in length, in which case, we conclude that the rectangles SC.SG, IC.IG have the same ratio as SG and IG, consequently the proportion

SR^2 : SC.SG :: IT^2 : IC.IG
will become SR^2 : SG :: IT^2 : IG;

that is, *the abscissas are as the squares of their corresponding ordinates*. But the same conclusion may be obtained thus :



Since $IT^2 = NI.IQ$, $SR^2 = HS.SK$ (by the prop. of the circle), and $NI = HS$, therefore IT^2 and SR^2 will have the same ratio as IQ and SK; but by similar triangles

$IG : SG :: IQ : SK :: IT^2 : SR^2$
(by equality);

That is, the squares of the ordinates IT and SR are in the same proportion as the abscissas IG and SG.

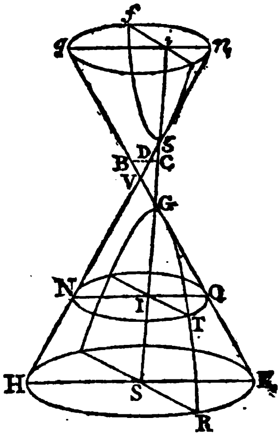
Put the absciss $SG = t$, the corresponding ordinate $SR = c$, and x and y for any other absciss and its ordinate:

Then $t : c^2 :: x : y^2$, whence $\frac{c^2}{t} x = y^2$, or, putting $\frac{c^2}{t} = p$, we have $px = y^2$, *the equation of a parabola*, where the constant quantity $\frac{c^2}{t}$, which is a third proportional to the axis and its conjugate, is called the *latus rectum*, or *parameter* of the axis.

269. Let the intersecting plane cut the opposite cones; and suppose $gi = GI$.

By similar triangles,
$IQ : iq :: GI : Gi$,
$IN : in :: gI : gi$,

whence $IQ.IN : iq.in :: GI.gi : Gi.gi$; but these latter rectangles are equal because $Gi = gI$, and $GI = gi$, consequently the two former must also be equal, or $IQ\ IN\ (= IT^2) = iq.in = if^2$, hence IT and if are equal, and therefore the opposite hyperbolas are similar and equal.



Again, by similar triangles,

$$SK : SG :: IQ : IG,$$
$$SH : Sg :: IN : Ig,$$

hence we have $SK.SH : SG.Sg :: IQ.IN : IG.Ig$: but $SK.SH = SR^2$, and $IQ.IN = IT^2$ (by prop. of circle),

and therefore $SR^2 : SG.Sg :: IT^2 : IG.Ig$,

that is, *the squares of the ordinates* (SR, IT) *have the same ratio as the rectangles of their corresponding abscissas* (SG, Sg, and IG, Ig), as in the ellipse.

If C be the center of the transverse axis Gg, and CB parallel to NQ or HK; then, by similar triangles,

$$IQ : IG :: CB : CG,$$
$$IN : Ig :: CD : Cg,$$

and $IQ.IN : IG.Ig :: CB.CD : CG.Cg$:

Now $IQ.IN = IT^2$, consequently the square of any ordinate (IT^2) and the rectangle of its corresponding abscissas ($IG.Ig$) have the constant ratio of the rectangle $CB.CD$ to the square of the semi-transverse ($CG.Cg$), hence a mean proportional between CB and CD will be the semi-conjugate to the transverse Gg. Let this be denoted by $\frac{1}{2}c$, and put $t = Gg$, $x =$ any absciss IG, and $y =$ the ordinate IT; then $t + x = Ig$ the other absciss (instead of $t - x$ as in the ellipse), and we shall have

$$y^2 : (t + x)x :: \tfrac{1}{2}c . \tfrac{1}{2}c : \tfrac{1}{2}t . \tfrac{1}{2}t,$$

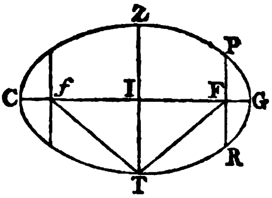
or $y^2 : tx + x^2 :: c^2 : t^2$; whence $\frac{c^2}{t^2}(tx + x^2) = y^2$, *the equation of an hyperbola.*

From the three equations thus obtained, we may derive the other principal properties of the curves by considering the sections in *plano* only, without any farther reference to the solid itself.

OF THE ELLIPSE.

270. SUPPOSE CG *and* ZT *are the transverse and conjugate axes, any absciss* FG $= x$, *and the corresponding ordinate* FR $= y$:

$$\textit{Then } \frac{ZT^2}{CG^2}(CG \times x - x^2) = y^2. \ (267)$$



Let the ordinate PR be a third proportional to the axes CG and ZT, that is $\frac{ZT^2}{CG} = PR$; then $\frac{ZT^2}{2CG} = FR$, and $\frac{ZT^4}{4CG^2} = y^2$;

therefore we have $\frac{ZT^2}{CG^2}(CG \times x - x^2) = \frac{ZT^4}{4CG^2}$,

or $CG \times x - x^2 = \frac{ZT^2}{4}$, this quadratic equation gives $x = \frac{1}{2}CG \pm \sqrt{\left(\frac{CG^2}{4} - \frac{ZT^2}{4}\right)}$, these two roots or values of x are the abscissas FC and FG;

$$\text{or } FC = IG + \sqrt{(IG^2 - IT^2)},$$
$$\text{and } FG = IG - \sqrt{(IG^2 - IT^2)}.$$

But $FG = IG - IF$, therefore $\sqrt{(IG^2 - IT^2)} = IF$, and consequently $FT = IG$.

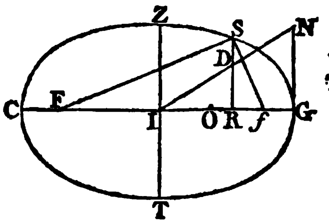
This ordinate PR, which is a third proportional to the axes, is called the *parameter* of the axis CG.

And if $If = IF$ then F and f are the *foci* of the ellipse.

Corol. 1. Hence Ff, the distance of the *foci*, is a mean proportional between the sum and difference of the axes. Or the distance of the *foci* from the center is a mean proportional between the sum and the difference of the semiaxes.

Corol. 2. And when the axes are given, the foci may be found by making TF, Tf, each = the semitransverse.

271. *If two lines are drawn from the foci to meet at any point in the curve, their sum will be equal to the transverse axis: that is,* $FS + fS = CG$.



Make GN perpendicular to IG, and = IZ the semiconjugate, and join IN; also let RS be an ordinate at right angles to IG.

Then the triangles IRD, IGN being similar, we have

$$IG^2 : GN^2 \ (IZ^2) :: IR^2 : RD^2,$$

whence $IG^2 : IZ^2 :: IG^2 - IR^2 : IZ^2 - RD^2$ (by alternation and division).

And $IG^2 : IZ^2 :: IG^2 - IR^2$ (or $\overline{IC + IR} . \overline{IG - IR}$) : RS^2, (267)

whence by equality $RS^2 = IZ^2 - RD^2$.

Now $FR = FI + IR$, and $FR^2 = FI^2 + 2FI \times IR + IR^2$:

but $FS^2 = FR^2 + RS^2$;

whence $FS^2 = FI^2 + 2FI \times IR + IR^2 + IZ^2 - RD^2$ (by addition),

or $FS^2 = FI^2 + IZ^2 + 2FI \times IR + IR^2 - RD^2$;

but $FI^2 + IZ^2 = IG^2$, (270. corol. 1)

whence $FS^2 = IG^2 + 2FI \times IR + IR^2 - RD^2$ (by substitution).

Let IO be a 4*th.* proportional to CG, Ff, and IR, that is, $2IG : 2FI :: IR : IO$; then $2FI \times IR = 2IG \times IO$ this substituted for $2FI \times IR$ in the last equation, and we have

$$FS^2 = IG^2 + 2IG \times IO + IR^2 - RD^2.$$

Again, since $2IG : 2FI :: IR : IO$, or $IG : IR :: FI : IO$,

we get $IG^2 : IR^2 :: FI^2$ (or $IG^2 - IZ^2$) : IO^2;

But from the similar triangles IRD, IGN,

$$IG^2 : GN^2\ (= IZ^2) :: IR^2 : RD^2,$$

whence $IG^2 : IG^2 - IZ^2 :: IR^2 : IR^2 - RD^2$ (by division),

or $IG^2 : IR^2 :: IG^2 - IZ^2 : IR^2 - RD^2$.

Also $IG^2 : IR^2 :: IG^2 - IZ^2 : IO^2$ (from the 4th. proportional), whence by equality, $IR^2 - RD^2 = IO^2$ this substituted for $IR^2 - RD^2$ in the latter of the preceding values of FS^2;

and the result is $FS^2 = IG^2 + 2IG \times IO + IO^2$;

and the square roots give $FS = IG + IO$ or $CI + IO$.

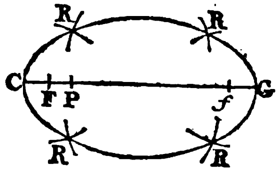
And by proceeding exactly in the same manner with $fR^2 = (fI - IR)^2$ instead of $FR^2 = (FI + IR)^2$, we shall get $fS = IG - IO$,

Therefore FS and fS are respectively equal to OC and OG, the two parts of the transverse diameter.

Corol. Hence is derived the common method of describing this curve with a thread, thus:

Let the thread be equal in length to the transverse CG, and fix its ends at the foci F and *f*, then move a pencil or pen round by the thread, keeping it always stretched, and the point of the pencil or pen will describe the curve.

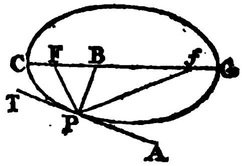
Or the curve may be traced mechanically, thus:



Take any point P in the transverse, and with PC, PG as radii, about the foci F, *f*, describe arcs intersecting each other in R, R, R, R which will be 4 points in the curve; and the like number may be found by assuming another point in CG, and so on: the curve is then to be drawn through the points of intersection.

272. *To draw a tangent to an ellipse at a given point* P *in the curve.*

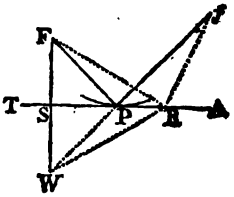
Let the point and the foci F, *f*, be joined, and bisect the angle FP*f* with PB; then TA drawn through P at right angles to PB, or to make the angle FPT =*f*PA, will be the tangent required.



The truth of this construction will be manifest from the following

THEOREM. *If from two given points* F, *f*, *two lines* FP, *f*P *be drawn to meet at a given right line* TA, *and make equal angles* FPT, *f*PA, *the lines* FP, *f*P *taken together will be less than any other two lines* FR, *f*R, *drawn from the same points to meet on the same line* TA.

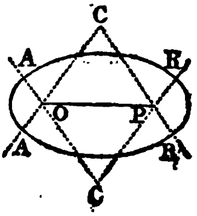
Draw FW at right angles to TA, and produce *f*P to meet it in W; also let WR be joined;



Then since the angle FPS =*f*PR = SPW, the right angled triangles SPF, SPW are similar and equal, and therefore FP = WP, and FP + *f*P = W*f*; but WR = FR, whence FR + *f*R = *f*R + RW which is greater than W*f* or its equal FP + *f*P.

The following method of drawing an Oval is frequently practised by workmen.

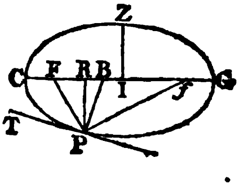
Two equal isoceles triangles OCP, OCP, are constructed on a common base OP, and the sides CO, CP produced; then C, C, are the centers of the circular arcs AR, AR (decribed with a pair of common compasses) and O, P, the centers of the arcs AA and RR.



Hence, if F and f be the foci of an ellipse, and $FP+fP$ equal to the transverse axis, it follows that an elliptical arc described with a thread (as directed in the preceding corollary) will pass through the point P and touch the line TA in that point: for if $FP+fP$ be *increased*, the curve thus described must cut the line in two points, but when *diminished*, it can neither intersect nor touch it, as is evident from the description of the curve.

It has already been observed (253) that opticians find when a ray of light falls on a reflecting surface, the angles of incidence and reflexion are equal; hence, if either focus be a lucid point, and the concavity of the ellipse a polished surface, the rays issuing from that point or focus will be reflected to the other: thus the angle $FPB = fPB$, and the ray FP is reflected in the direction Pf; hence it is, that F, f, are called *foci* or *burning points*.

273. *Let* TP *be a tangent to the ellipse,* F, f, *the foci,* I *the center, and* PR *an ordinate to the axis* CG; *then* $IC^2 : IF^2 :: IR : IB$.



Since PB bisects the angel FPf we have $fP : FP :: fB : FB$ (Geom. 96) whence $fP + FP : fP - FP :: FB + fB : fB - FB$, (by composition and division)

$$\text{or } 2IC : fP - FP :: 2IF : 2IB,$$

$$\text{or } IC : \frac{fP - FP}{2} :: IF : IB,$$

$$\text{and } IC^2 : \frac{fP - FP}{2} \times IC :: IF : IB. \ (92)$$

But in the triangle FPf, I is the middle of the base Ff, PR the perpendicular on that base, and $fP+FP$ (or $2IC$) is the sum of the sides;

whence (245) $Ff : fP + FP$ (or $2IC$) $:: fP - FP : 2IR$,

and $\frac{Ff}{2}$ (or IF) $: IC :: \frac{fP - FP}{2} : IR$,

whence $\frac{fP - FP}{2} \times IC = IF \times IR$:

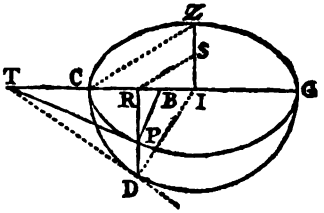
Therefore $IC^2 : IF \times IR :: IF : IB$ (by substitution),
or $IC^2 : IF \times IR :: IF^2 : IB \times IF$, (92)
and $IC^2 : IF^2 :: IF \times IR : IB \times IF$ (by alternation),
that is $IC^2 : IF^2 :: IR : IB$.

Corol. Let IZ be the semiconjugate axe;
then since $IC^2 : IF^2 :: IR : IB$,
we have $IC^2 : IC^2 - IF^2 :: IR : IR - IB$ (or BR), by division,
that is $IC^2 : IZ^2 :: IR : BR$, because $IC^2 - IF^2 = IZ^2$. (270)

274. *If* TP *be a tangent,* CG *the transverse axis,* IZ *the semiconjugate,* PB *perpendicular to* TP, *and* PR *an ordinate; then* IC *is a geometrical mean between* IR *and* IT.

Let RS be parallel to CZ.

Then $IC^2 : IZ^2 :: IR^2 : IS^2$ (by sim. triang.)
But $IC^2 : IZ^2 : IR : BR$ (273. corol.) $:: IR^2 : IR \times BR$;
whence $IR^2 : IR \times BR :: IR^2 : IS^2$ (by equality),
and therefore $IR \times BR = IS^2$.



Moreover, the right angled triangles TRB, PRB being similar,
we have $TR : RP :: RP : BR$,
whence $TR \times BR = RP^2$.

But $IC^2 : IZ^2 :: IC^2 - IR^2 : IZ^2 - \frac{IR^2 . IZ^2}{IC^2} = RP^2$. (267).

And by sim. triang. $IC^2 : IZ^2 :: IR^2 : \frac{IR^2 . IZ^2}{IC^2} = IS^2$,

whence, by substitution $IZ^2 - IS^2 = RP^2 = TR \times BR$,
or $IZ^2 = IR \times BR + TR \times BR = (IR + TR) BR = BR \times IT$.

And since BR : IR :: BR $\times$ IT : IR $\times$ IT, (92)
that is BR : IR :: IZ^2 : IR $\times$ IT (by substitution):
Also BR : IR :: IZ^2 : IC^2, (273 corol.)
therefore $IC^2 = IR \times IT$; that is, IC is a mean proportional between IR and IT.

Corol. 1. If TD be a tangent to a circle described on the transverse CG, and the points D and I joined, also suppose DR to be perpendicular to CG: then the angle TDI being a right one, we have, by sim. triangles, TI : DI (or IC) :: DI : IR, therefore IC is also a mean proportional between IR and IT in the circle, consequently the point D is in RP produced. Hence it follows, that if any number of ellipses have the same transverse (CG), the tangents drawn from the points (P, &c.) where an ordinate (RD) intersects them, will all meet in the same point (T) in the transverse produced. For IR and IT remain the same, whatever be the length of the conjugate.

Corol. 2. Since $IC^2 : IZ^2 : RC \times RG : RP^2$, (267) in the ellipse;

and $RD^2 = RC \times RG$ in the circle,
it will be $IC^2 : IZ^2 :: RD^2 : RP^2$,
or IC : IZ :: RD : RP;

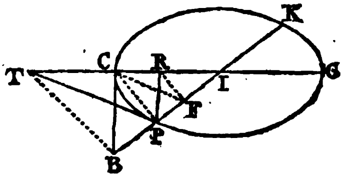
that is, the ordinates RP, RD, have always the same ratio as the semiaxes, or axes of the ellipse:

or *transverse* : *conjugate* :: RD : RP.

Corol. 3. Hence also, *the area of the ellipse will be a geometrical mean between the circles described on the two axes.* For if ordinates (RD, &c.) be drawn from every point in CG, the sum of all the RD's constitutes the area of the semicircle CDG, and the sum of all the RP's that of the semiellipse CPG; but the sum of the former to that of the latter is as RD to RP, or as the transverse axe to the conjugate. And any corresponding segments have evidently the same proportion.

Hence, if t be put to denote the transverse, and c the conjugate, then $.7854t^2$ is the area of the circle described on the transverse, and $t : c :: .7854t^2 : .7854tc$ the area of the ellipse, which is a mean proportional between the two areas $.7854t^2$ and $.7854c^2$.

275. *If* CG *be the transverse axis,* TP *and* CB *tangents at* P *and* C, *respectively, and* KPB *a diameter produced; then the triangles* IPT, ICB *will be equal.*



Draw the ordinate PR, and let CF be parallel to TP.

Then the triangles PRI, BCI; CFI, TPI being respectively similar, we have

IF : IP :: IC : IT :: IR : IC (274) :: IP : IB, (by sim. triang.)

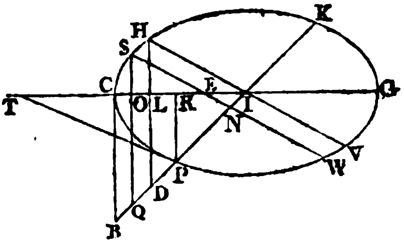
that is, IF : IR :: IP : IC,
and IC : IP :: IT : IB

therefore (Geom. 94. *corol.* 1) RF, CP, TB are parallel to each other.

Now the triangles TCP, BCP on the same base CP, and between the parallels CP, TB, are equal, therefore adding CPI to each we have IPT = ICB.

Corol. The triangle PRT = trapezoid CRPB; this appears by taking the triangle PRI from each of the triangles IPT, ICB.

276. *Let the diameter* HV *be parallel to the tangent* TP, *then* PK *and* HV *are called conjugate diameters. And if* SN *be an ordinate to* PK, *and* HD, SQ *parallel to the tangent* CB, *the triangle* SOE = *trapezoid* COQB.



Let PR be an ordinate to CG:

By similar triangles,
$IC:CB::IR:RP::IO:OQ$,
whence by alternation and division,

$IC : CB :: IC+IR : CB+RP :: IC+IO : CB+OQ$,

or $IC+IO : IC+IR :: CB+OQ : CB+RP$ (by alternation).

But

$OC\times OG=OC.(IO+IC)$, and $RC\times RG=RC.(IR+IC)$; whence

$OC\times OG : RC\times RG :: OC.(IO+IC) : RC.(IR+IC)$,

or

$OC\times OG : RC\times RG :: OC.(CB+OQ) : RC.(CB+RP)$ by equality;

$$:: OC.\left(\frac{CB+OQ}{2}\right) : RC.\left(\frac{CB+RP}{2}\right).$$

:: *trapez.* COQB : *trapez.* CRPB.

Moreover, the triangles PRT, SOE being similar, we have *triang.* SOE : *triang.* PRT :: $SO^2 : RP^2 :: OC\times OG : RC\times RG$, (267)

or *triang.* SOE : *triang.* PRT :: *trapez.* COQB : *trapez.* CRPB, (by equality):

But (275 corol.) *triang.* PRT = *trapez.* CRPB, therefore, by equality, the *triang.* SOE = *trapez.* COQB. And in the same manner it is proved that the *triangle* HLI = *trapez.* CLDB.

Corol. From the equal triangles IPT, ICB, take the triang. EIN, then the trapez. TPNE = quadrilateral BCEN; from this subtract the trapez. COQB, and add its equal, or the triang. SOE, to the remainder QOEN, and we have the triang. SQN = trapez. TPNE. In like manner, by subtracting CLDB from the triang. ICB, and adding its equal HLI, we get the triang. HDI = triang. ICB = triang. IPT.

277. *If* PK *and* HV *be conjugate diameters, and* SN *an ordinate, as in the preceding Theorem,*

Then $IP^2 : IH^2 :: NP \times NK : NS^2$.

By similar triangles,

triang. IPT : *triang.* $EIN :: IP^2 : IN^2$;

and $IPT : IPT - EIN :: IP^2 : IP^2 - IN^2$ (by division),

that is, *triang.* IPT : *trapez.* $TPNE :: IP^2 : (IP + IN) . (IP - IN)$, or $NK \times NP$.

Moreover, since the triangles HDI, SQN are similar, we have

$IH^2 : NS^2 ::$ *triang.* DHI : *triang.* SQN;

that is $IH^2 : NS^2 ::$ *triang.* IPT : *trapez.* $TPNE$ (by equality)

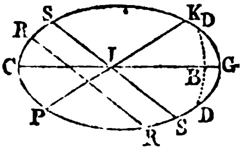
$:: IP^2 : NK \times NP$;

or alternately, $IP^2 : IH^2 :: NK \times NP : NS^2$. And in the same manner it is proved that $IP^2 : IV^2 :: NK \times NP : NW^2$: therefore $NW = NS$.

Corol. 1. Hence any diameter bisects all its double ordinates.

Corol. 2. And the property demonstrated in regard to the transverse axis (Art. 267) is general for any diameter whatever, viz. *the rectangles of the segments of any diameter, are as the squares of their corresponding ordinates.*

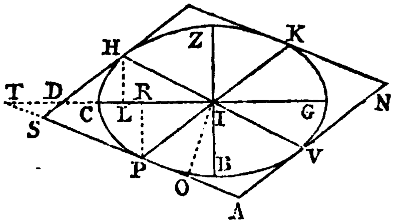
Corol. 3. Hence also is derived the method of finding the center and axes of an ellipse; thus,



Draw two parallel ordinates or lines SS, RR, and bisect them with PK which will be a diameter; then about I the center, or middle of PK, describe an arc of a circle meeting the ellipse in two points DD, bisect DD in B, and through B, I, draw GC which will be the transverse. If the arc falls without the ellipse it gives the conjugate.

278. *Every parallelogram circumscribing an ellipse is equal to the rectangle of the two axes.*

Let PK, HV be conjugate diameters, then the tangents through their extremities will form a parallelogram SN. (276).



Draw the axes CG, ZB, and the ordinates PR, HL, also suppose GC, AS are produced till they meet in T, and let IO be perpendicular to SA.

Then (274) $IC^2 = IR.IT$, and $IC^2 = IL.ID$;
therefore $ID : IT :: IR : IL$.

But the triangles IHD, TPI; IHL, TPR are respectively similar,
whence $ID : IT :: IL : TR$,
therefore by equality, $IR : IL :: IL : TR$, or $IL^2 = IR.TR$:
Also, because $IC : IR :: IT : IC$, (274)
we have $IC - IR$ (or RC) : $IR :: IT - IC$ (or CT) : IC, by division,
or alternately, $IR : IC :: RC : CT$;
and $IR + IC$ (or RG) : $IR :: RC + CT$ (or RT) : RC, by composition,
or alternately, $IR : RG :: RC : RT$, hence $IR.TR = IL^2 = RG.RC$.

Moreover, because $IC^2 : IZ^2 :: RC.RG : RP^2$, (267)
or $IC^2 : IZ^2 :: IL^2 : RP^2$ (by equality),
we have $IC : IZ :: IL : RP$.
In like manner, since $IC^2 : IZ^2 :: LC.LG : LH^2$,
we shall get $IC : IZ :: IR : LH$,
or $IZ : LH :: IC : IR$ (by alternation).
But (274) $IT : IC :: IC : IR$,
therefore by equality $IT : IC :: IZ : LH$, or $IT.LH = IC.IZ$.
But the triangles IOT, ILH being similar, we have
$IH : IT :: LH : IO$, or $IT.LH = IH.IO$,
and therefore the rectang. $IH.IO = IC.IZ$, and $4IH.IO = 2IC.2IZ$.

But the rectang. $IH . IO$ is $=$ the parallelogram $PIHS$ or $\frac{1}{4}$ of the circumscribing parallelogram SN, consequently $2IC . 2IZ$ or the rectang. $CG . ZB =$ the parallelogram SN.

279. *The sum of the squares of any two conjugate diameters, is equal to the sum of the squares of the two axes. That is,* $PK^2 + HV^2 = CG^2 + ZB^2$: (see the preceding fig.)

By the last Theo. $IL^2 = RG . RC = IC^2 - IR^2$ (or $\overline{IC + IR} \times \overline{IC - IR}$),

therefore $IL^2 + IR^2 = IC^2$.

But $IC^2 : IZ^2 :: IL^2 : RP^2$,

and $IC^2 : IZ^2 :: IR^2 : LH^2$,

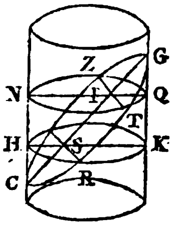
whence $2IC^2 : 2IZ^2 :: IL^2 + IR^2 : RP^2 + LH^2$, (by composition)

that is $2IC^2 : 2IZ^2 :: IC^2 : RP^2 + LH^2$;

therefore by equality $RP^2 + LH^2 = IZ^2$:

Hence the sum of the 4 squares $IL^2 + IR^2 + RP^2 + LH^2 = IC^2 + IZ^2$ the sum of the squares on the semiaxes: but (Geom. 83) the sum of those 4 squares are equal to $IH^2 + IP^2$ the squares on the semiconjugates; therefore $4IH^2 + 4IP^2 = 4IC^2 + 4IZ^2$, or $HV^2 + PK^2 = CG^2 + ZB^2$.

280. *If a cylinder be cut by a plane oblique to its axis, the section is an ellipse.*



Let CZGT be the section, HRK, NTQZ, two sections parallel to the base of the cylinder. Then the equation of the curve is derived exactly as in the cone, (267); thus

By similar triangles,

$SC : SH :: IC : IN$,

$SG : SK :: IG : IQ$,

whence (96) $SC . SG : SH . SK$ (or SR^2) $:: IC . IG : IN . IQ$ (IT^2 or IZ^2),

or $SR^2 : SC . SG :: IT^2 : IC . IG$,

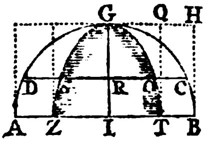
That is, *the squares of the ordinates are as the rectangles of the corresponding abscissas.*

The conjugate axis ZT is the diameter of the cylinder, I being the center of the transverse CG.

If $CG = t$, $ZT = c$, SC or $SG = x$, $SR = y$, then the foregoing proportion gives $\frac{c^2}{t^2}(tx - x^2) = y^2$ *the equation of the ellipse.*

281. *The spheroid or solid generated by the revolution of an ellipse about either axis, is $\frac{2}{3}$ of the circumscribing cylinder.*

Let ZTG be a semiellipse, ZT the conjugate axis, IG the semitransverse, and AGB a semicircle described about the center I.



Then if the ellipse and circle revolve about the axis IG, the former will describe a hemispheroid, and the latter an hemisphere.

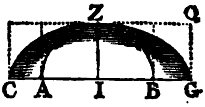
Suppose DC to be a plane parallel to the base AB; then SO and DC will be the diameters of the circular sections of the two solids made by that plane. And because AB : ZT :: DC : SO at every point in IG (274) the surfaces of the corresponding circular sections will be in the constant ratio of AB^2 to ZT^2: if therefore we conceive the two solids to be composed of an infinite number of indefinitely thin elementary circular parallel planes, (Geom. 131) the sum of those in the hemisphere AGB will be to the sum of those in the hemispheroid ZGT, as AB^2 to ZT^2,

that is, $AB^2 : ZT^2$ :: *solid* AGB : *solid* ZGT:

But if AH, ZQ are the cylinders circumscribing the two solids, then $AB^2 : ZT^2$ :: *cylind.* AH : *cylind.* ZQ, (Geom. 131. *corol.* 3)

therefore by equality, *solid* AGB : *solid* ZGT :: *cylind.* AH : *cylind.* ZQ :

But (Geom. 134) the solid or hemisphere AGB $= \frac{2}{3}$ of the cylinder AH, consequently (by the proportion) the solid or hemispheroid ZGT $= \frac{2}{3}$ of the cylinder ZQ.

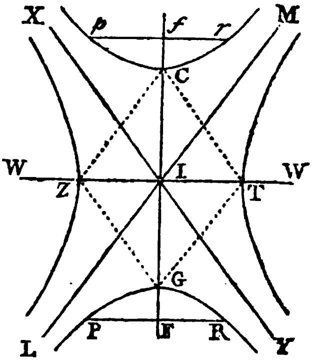


And if the ellipse, and the semicircle described about I with the radius IZ, revolve about the semiconjugate axis IZ, then, in the same manner, it is proved that half the spheroid (or the solid CZG) is equal to $\frac{2}{3}$ of the circumscribing cylinder CQ. This is called an *oblate* spheroid, or ellipsoid. But when the transverse diameter of the ellipse is the fixed axis, the solid is a *prolate* spheroid.

Corol. $AB^2 : ZT^2$ (or $DC^2 : SO^2$) :: *spherical segment* DGC : *spheroidal segment* SGO.

OF THE HYPERBOLA.

282. THE equation of the curve, or fundamental property, is already derived from the cone (269); but in considering the section in *plano*, the following definitions will be necessary.



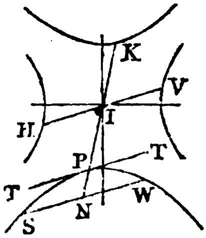
1. If PGR, pCr be the opposite hyperbolas, GC the transverse axis, I its center, F, f, the foci, WIW at right angles to CG, and GZ, GT, CZ, CT, each equal to IF or If, then ZT is the conjugate axis.

2. When the conjugate axis is equal to the transverse, or ZT =CG the curve is called an equilateral hyperbola, or right angled hyperbola.

3. The line PFR at right angles to the axis CF, is called the *parameter*, or *latus rectum.*

4. Two indefinite right lines LM, XY, drawn through the center I parallel to GT, GZ, are called the *asymptotes* of the hyperbola, or of the opposite hyperbolas.

5. If ZT, CG, are made the transverse, and conjugate to two other hyperbolas, whose vertices are Z and T, those are called *conjugate hyperbolas*, with respect to the former.



6. Any right line PK drawn through the center I, and terminated at the opposite sections, is called a *diameter*, and the extremities P, K, its vertices: and HV parallel to TT the tangent at P, is called its *conjugate diameter*.

7. If any diameter KP be continued within the curve, the inner part PN is called the abscissa; and SW parallel to the tangent TT is a double ordinate to the diameter KP.

283. *The square of the distance of the focus from the center is equal to the sum of the squares of the semiaxes: viz.* $If^2 = IG^2 + IZ^2$. (see the following *fig.*)

Let IZ be the semiconjugate, fP the semiparameter or third proportional to the semitransverse IG and semiconjugate IZ:

Then the equation of the curve (269) gives

$IG^2 : IZ^2 :: (If + IG) . (If - IG)$ or $If^2 - IG^2 : fP^2$:

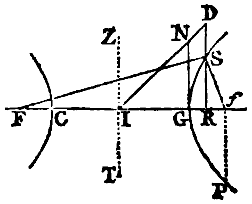
But $IG : IZ :: IZ : fP$ (the semiparam.)

and $IG^2 : IZ^2 :: IZ^2 : fP^2$,

therefore by equality $IZ^2 = If^2 - IG^2$, or $IZ^2 + IG^2 = If^2$; instead of $IG^2 - IZ^2 = If^2$ as in the ellipse, (270 corol. 1).

Corol. Hence the distance from Z to G set on the axis from the center I both ways, give the foci f, F.

284. *If two lines are drawn from the foci to meet at any point in the curve, their difference will be equal to the transverse axis: that is* $FS - fS = CG$.



Make GN perpendicular to IG, and = IZ the semiconjugate, and join IN; also let RS be an ordinate at right angles to IG.

Then RS being produced to D, the triangles IRD, IGN, will be similar, and we shall have

$$IG^2 : GN^2\ (IZ^2) :: IR^2 : RD^2,$$

whence $IG^2 : IZ^2 :: IR^2 - IG^2 : RD^2 - IZ^2$ (by alternation and division).

And $IG^2 : IZ^2 :: (IR + IG) . (IR - IG)$ or $IR^2 - IG^2 : RS^2$, (269)

whence by equality $RS^2 = RD^2 - IZ^2$:

Now $FR = FI + IR$, and $FR^2 = FI^2 + 2FI \times IR + IR^2$,

but $FS^2 = FR^2 + RS^2$,

whence $FS^2 = FI^2 + 2FI \times IR + IR^2 + RD^2 - IZ^2$ (by addition),

or $FS^2 = FI^2 - IZ^2 + 2FI \times IR + IR^2 + RD^2$:

But $IG^2 = FI^2 - IZ^2$, (283)

whence $FS^2 = IG^2 + 2FI \times IR + IR^2 + RD^2$.

Let IO be a fourth proportional to CG, Ff, and IR, that is $2IG : 2FI :: IR : IO$; then $2FI \times IR = 2IG \times IO$, this substituted for $2FI \times IR$ in the last equation, and we have

$$FS^2 = IG^2 + 2IG \times IO + IR^2 + RD^2:$$

Again, since $2IG : 2FI :: IR : IO$, or $IG : IR :: FI : IO$, we get $IG^2 : IR^2 :: FI^2$ (or $IG^2 + IZ^2$) $: IO^2$.

But from the similar triangles IRD, IGN,

$$IG^2 : GN^2\ (= IZ^2) :: IR^2 : RD^2;$$

whence $IG^2 : IR^2 :: IG^2 + IZ^2 : IR^2 + RD^2$ (by composition);

Also $IG^2 : IR^2 :: IG^2+IZ^2 : IO^2$ (from the 4th. proportional),

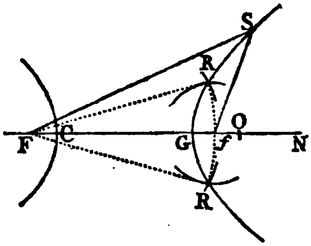
whence, by equality, $IR^2 + RD^2 = IO^2$ this substituted for IR^2+RD^2 in the latter of the preceding values of FS^2,

and the result is $FS^2 = IG^2 + 2IG \times IO + IO^2$;

and the square roots give $FS = IG + IO$.

And proceeding in the same manner with $fR^2 = (fI - IR)^2$ instead of $FR^2 = (FI+IR)^2$, we shall get $fS = IO - IG$;

Therefore $FS - fS$ or $IG + IO - (IO - IG) = 2IG = CG$ the *difference* of the two lines drawn from the foci to meet in the curve. In the ellipse their *sum* is $=$ the transverse CG. (271)

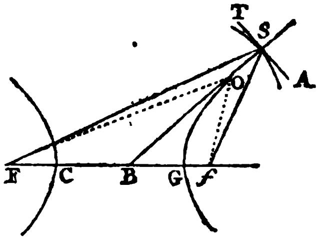


Corol. Hence is derived a method of describing the curve by continued motion when the transverse and foci are given; thus,

Let two threads FS, fS, whose difference in length is $=$ the transverse CG, be fastened at the foci F, f; then if the other ends are tied together (suppose at S) and passed through a small bead or pin S, and the bead or pin be made to move along the threads while they are constantly kept tight, the said bead or pin will, by its motion, describe the curve.

Or the curve may be traced mechanically, thus, Take any point O in the axis fN, then with GO and CO as radii, about the foci f, F, describe arcs of circles intersecting each other in R, R, which will be two points in the curve: and the like number may be found by assuming another point in the axis (fN), and so on. The curve is then to be drawn through the points of intersection.

285. *To draw a tangent to the hyperbola at a given point S in the curve.*

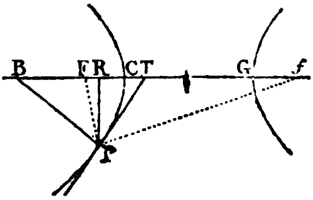


Let SF, Sf be drawn to the foci F, f; then a line SB which bisects the angle FSf is the tangent required.

Suppose TA is drawn through S to make the angles FST, fSA, equal; then if F, f, were the foci of an ellipse, and the length of the threads $SF + Sf$ remained constant, an elliptical arc might be described with that constant length which would touch TA in the point S, and SB would bisect the angle FSf, or stand at right angles to TA (272); but in describing the hyperbola, the threads SF, Sf, are constantly diminished equally in length, and consequently the motion of the point S must, for that reason, be at right angles to the direction of the curve of the ellipse at that point, that is, an indefinitely small part of the hyperbolic curve (SO) will coincide with SB, which therefore must be a tangent to the curve.

Corol. Hence if TA were a reflecting surface perpendicular to the curve, a ray of light FS proceeding from one focus, would be reflected in the direction Sf to the other.

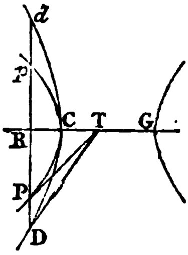
Scholium. In comparing Articles 271 and 280, the reader will perceive that the latter may be considered as a repetition of the former; the difference consisting merely in the signs + and —, which vary in a few steps of the process. This similarity extends to most of the properties of the Ellipse and Hyperbola. We therefore shall only enunciate the theorems in the three following articles, and leave their investigations as exercises for the student, who will find little difficulty in framing the demonstrations when he comprehends what is laid down respecting the ellipse.



286. Let TP be a tangent to the hyperbola, F, *f*, the foci, I the center, and PR an ordinate to the axis CG.

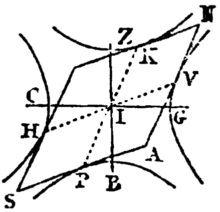
Then $IC^2 : IF^2 :: IR : IB$, (PB being perpendicular to the tangent, as in the ellipse, *art.* 273).

And IC is a geometrical mean between IR and IT; that is, $IR : IC :: IC : IT$.



Hence also, if two or more hyperbolas have the same common axis CG, the tangents at the extremities of the ordinates RP, RD, &c. will all meet in the same point T in the axis, as in the ellipse, (274 *corol.* 1.)

And the ordinates RP, RD have the same ratio as the conjugate axes of the hyperbolas. Whence it follows that the hyperbolic spaces PC*p*, and DC*d*, are also proportional to those axes; for each is composed of the like indefinite number of parallel ordinates whose sums are respectively as RP to RD.



287. Every parallelogram inscribed between the four conjugate hyperbolas is equal to the rectangle of the two axes:

That is, the parallelogram $SN = CG \times ZB$.

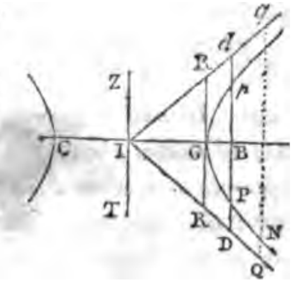
And the opposite sides are bisected at the points of contact H, K, V, P, as in the ellipse (*art.* 278).

288. The difference of the squares of any two conjugate diameters, is equal to the difference of the squares of the two axes:

That is, $HV^2 - PK^2 = CG^2 - ZB^2$. In the ellipse their *sums* are equal, (*art.* 279.)

OF THE HYPERBOLIC ASYMPTOTES.

289. *If* I *be the center of the hyperbola,* CG *the transverse axis,* RR ($=$ ZT) *the conjugate,* ID, I*d the asymptotes,* P*p an ordinate produced to* D *and d: Then* $Pd \times PD = GR^2$. (RR *being the tangent at* G).



By similar triangles, $IB^2 : BD^2 :: IG^2 : GR^2$ (or IZ^2):

And (269) $IG^2 : GR^2 :: (CG + GB) \times GB$ or $(IB + IG)(IB - IG)$, or $IB^2 - IG^2 : BP^2$;

That is $IB^2 : BD^2 :: IB^2 - IG^2 : BP^2$ (by equality);

or alternately

$$IB^2 : IB^2 - IG^2 :: BD^2 : BP^2;$$

and $IB^2 : IB^2 - (IB^2 - IG^2) :: BD^2 : BD^2 - BP^2$ (by division);

That is $IB^2 : IG^2 :: BD^2 : BD^2 - BP^2$,

or alternately $IB^2 : BD^2 :: IG^2 : BD^2 - BP^2$;

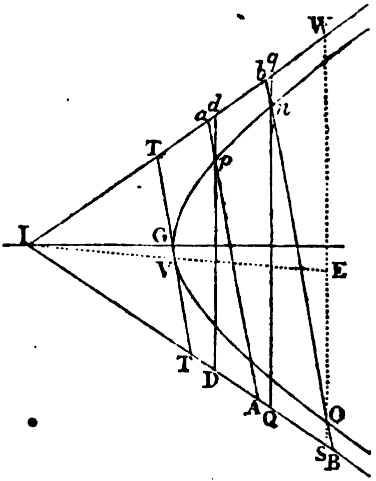
whence $IG^2 : GR^2 :: IG^2 : BD^2 - BP^2$ (by equality);

Therefore $GR^2 = BD^2 - BP^2 = (BD + BP)(BD - BP)$ or $Pd \times PD$.

Corol. Hence if Q*q* be any other parallel ordinate produced, then $Nq \times NQ = Pd \times PD$: for each is $= GR^2$.

290. *If two parallel lines* A*a,* B*b are drawn through the hyperbola to meet the asymptotes; then* $pA \times pa = nB \times nb$.

Through *p* and *n* draw ordinates to the axis: Then the triangles *p*DA, *n*QB; *pda*, *nqb*, will be respectively similar;



whence

$pD : pA :: nQ : nB$,

and $pd : pa :: nq : nb$,

therefore

$pD.pd : pA.pa :: nQ.nq : nB.nb$ (by compounding):

But (289 corol.)

$pD.pd = nq.nq$;

therefore

$pA.pa = nB.nb$.

291. *If any right line* (Bb) *be drawn through the hyperbola to meet the asymptotes; then the parts of that line between the curve and asymptotes will be equal: that is* $OB = nb$.

Let WOS be parallel to qQ or dD. Then the triangles QnB, SOB, and also bnq, bOW being respectively equal,

we have $nQ : OS :: nB : OB$,

and $nQ : OW :: nb : Ob$,

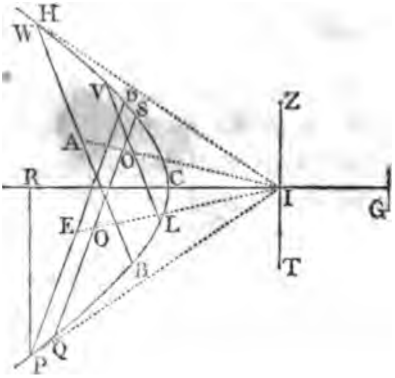
whence $nQ.nQ : OS.OW :: nB.nb : OB.Ob$ (by compounding):

But (289 corol.) $nQ.nq = OS.OW$, therefore $nB.nb = OB.Ob$; whence it follows that $OB = nb$.

Corol. 1. If the tangent TT be parallel to bB, then it is bisected in the point of contact V; for if Bb be supposed to move parallel to itself towards I, the points O, n, will coincide at V. In which case the rectangle $OB.Ob$ becomes $VT.VT$ or VT^2. Therefore if V be the vertex of the semidiameter VI, TT a tangent at that point, and aA a line parallel to TT, then $pA.pa = VT^2$.

Corol. 2. Hence also it appears, that if the semidiameter IV be produced, it will bisect all the corresponding double ordinates, On, &c. for since TT is bisected in V, it follows from similar triangles, that its parallels aA, bB, &c. are bisected, and because nb = OB, the double ordinates nO, &c. are also bisected by VE.

Corol. 3. Hence when the curve of an hyperbola is given, the axes may be determined thus,



Draw parallel ordinates VL, WB, and SQ, DP, and let them be bisected in O, A, and O, E by the lines AI, EI, then their point of concourse I will be the center. Take any two points P, H, in the curve equally distant from the center I, and bisect the angle PIH with the line ICR; then if IG = IC, CG will be the transverse axe.

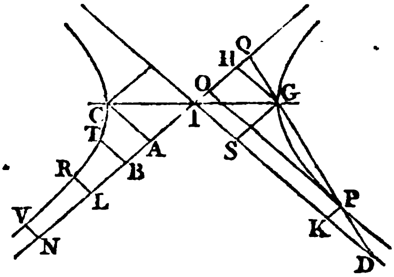
From any point P in the curve, draw an ordinate PR to the axis CG,

$$\text{then (269) } GR \times CR : PR^2 :: CG^2 : (\textit{conjug. axis})^2,$$

$$\text{or } \sqrt{(GR \times CR)} : PR :: CG : \textit{conjug. axis};$$

That is, the conjugate axis ZT is a fourth proportional to the mean proportional between GR and CR, the corresponding ordinate PR, and the transverse axis CG.

292. *If* P *be any point in the curve, and* PK, GS; PO, GH, *parallel to the asymptotes* IQ, ID, *respectively; then the parallelograms* PKIO, GSIH *are equal.*



Through G and P draw QD: Then because PK and KD are parallel to QH and HG, and $GQ = PD$ (291), we shall have $HG = KD$, and $HQ = KP = IO$, therefore $OQ = IH = HG = GS$.

And by sim. triang.

$$HQ : HG :: OQ \text{ (or } GS) : OP;$$

whence $HQ \times OP = HG \times GS$: but $KP = HQ$, therefore $KP \times OP = HG \times GS$.

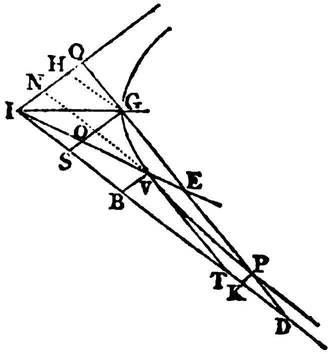
Now as the rectangles of the sides of the parallelograms SH, KO, about the equal angles KPG, SGH are equal, it follows (from Art. 258 *Mensur.*) that the parallelograms themselves must also be equal.

Corol. 1. Hence all the inscribed parallelograms are equal to one another: for each is equal to SGHI.

Corol. 2. It also appears that *the asymptote continually approaches towards the curve, but can never meet it:* for $KP \times OP = HG \times GS$ (*a given magnitude*), consequently $\frac{HG \times GS}{OP} = KP$, which must always be of some length if OP is assignable. Thus suppose HG, GS are each an *inch*, and $OP = 10000$ *miles*, then $KP = \frac{1}{10000 \times 1760 \times 36}$ of an *inch*. The distance of the curve and asymptote therefore diminishes as the latter is increased; on which account, the asymptote is sometimes considered as a tangent to the curve at an infinite distance.

Corol. 3. If IA, IB, IL, IN, &c. are in geometrical progression ascending, then AC, BT, LR, NV, &c. (parallel to the other asymptote) will be a descending progression: for the rectangles $IA \times AC$, $IB \times BT$, &c. being equal, BT, LR, NV, &c. are reciprocally as IB, IL, IN, &c. Thus if $IA = 1$, $IB = \frac{3}{2}$, $IL = \frac{9}{4}$; $IN = \frac{27}{8}$, &c. then $AC = 1$, $BT = \frac{2}{3}$, $LR = \frac{4}{9}$, $NV = \frac{8}{27}$, &c.

293. *Let* TV *be a tangent at* V, *and* DQ *parallel to that tangent; then if* GS, VB, PK, *are parallel to the asymptote* IQ, VB *will be a geometrical mean between* GS *and* PK; *that is* $VB^2 = PK \times GS$.



The triangles KPD, BVT, SGD being similar, we have

$VB : VT :: PK : PD$,

$VB : VT :: GS : GD$,

whence

$VB^2 : VT^2 :: PK \times GS : PD \times GD$ (by compounding);

But

$VT^2 = PD \times PQ$ or $PD \times GD$, (291 *cor.* 1)

Therefore by equality, $VB^2 = PK \times GS$.

294. *The mixt-lined quadrilinear spaces* GVBS, VPKB *are equal.*

Since VE, the diameter produced, bisects all the double ordinates (291 corol. 2) each of the spaces EVG, EVP is composed of an infinite number of equal ordinates EG, EP, &c. therefore by the *method of indivisibles*, those spaces are equal.

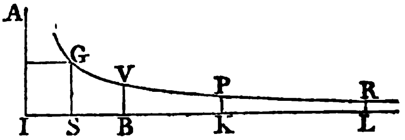
And because the triangles PKD, QHG are similar, and $PD = GQ$, those triangles are also equal:

Now QD is bisected in E, consequently the triangle EDI = triang. EQI:

From the triang. EDI take EVP + PKD, and from EQI subtract EVG + QHG, and we have the space VPKI = space

VGHI: and if the equal triangles VBI, VNI are respectively subducted from those equal spaces, then the space VPKB = space VGHN:

But (292) the parallelogram BVNI = SGHI; from each of these take the common parallelogram SONI, and we have OVBS = OGHN, and if to each we add the trilinear OGV, there results GVBS = VGHN = VPKB.



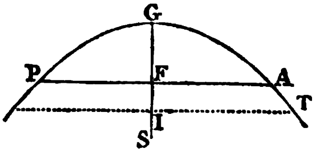
SCHOLIUM. The asymptotic space GVPKS is therefore bisected by the ordinate or line VB which is a geometrical mean between the extremes GS, PK. Hence it appears that when GS, VB, PK, RL, &c. are in geometrical progression, the included spaces GVBS, VPKB, PRLK, &c. are equal; and the spaces GVBS, GPKS, GRLS, &c. proceed in arithmetical progression, while the corresponding distances IB, IK, IL, on the asymptote, are in geometrical progression: the former are therefore analogous to the *logarithms* of the latter. Thus suppose the hyperbola is equilateral or the asymptotes IA, IL are at right angles, and GS = IS = 1, IB = 2, IK = 4, IL = 8, &c. then the area of the space GVBS = 0.693147 the *log.* of 2 or IB; the area GPKS = 1.386294 the *log.* of 4 or IK; the area GRLS = 2.079441 the *log.* of 8 or IL, &c. These logarithms are called *hyperbolic logarithms.*

The system of logarithms however, will vary with the angle made by the asymptotes: Thus if they form an angle of $25^\circ\ 44'\ 25''\tfrac{1}{2}$, and IS=GS=1, IB=2, IK=4, IL=8, &c. the area of the rhombus GI will be 0.4342914819; and the asymptotic spaces GVBS, GPKS, GRLS, &c. equal to 0.30103, 0.60206, 0.90309, &c. respectively, which are *Briggs's* logarithms of 2, 4, 8, &c. The area of the rhombus, or which is the same thing, that of any inscribed parallelogram, is called the *modulus* of the system.

OF THE PARABOLA.

295. LET G *be the vertex,* F *the focus,* GS *the axis; then the ordinate* PA *at right angles to* GS, *is the parameter. And* FG *the distance of the focus from the vertex is* $= \frac{1}{4}$PA.

Draw IT parallel to FA: Then from the equation of the curve, (268)

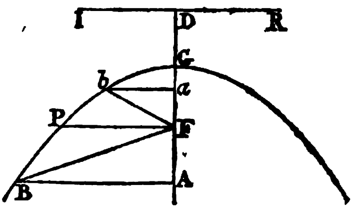


$$GI : IT^2 ; GF : \frac{IT^2 . GF}{GI} = FA^2, \text{ or } \frac{4IT^2 . GF}{GI} = PA^2.$$

But $GI : IT :: IT : \frac{IT^2}{GI}$ is the parameter $= PA$, by hypothesis, (268)

Therefore $\frac{IT^4}{GI^2} = \frac{4IT^2 . GF}{GI}$, or $\frac{IT^2}{GI} = 4GF$, that is, $PA = 4GF$.

296. *Let a line be drawn from the focus to any point* (B) *in the curve, and an ordinate* (BA) *from that point to the axis; also let* GD (*in the axis produced*) *be taken* = GF; *then* FB = DA.



Because $FA = GA - GF$, therefore $FA^2 = GA^2 - 2GA \times GF + GF^2$:

But (268) $BA^2 = p.GA = 4GF \times GA$ (p being the parameter),

whence $FA^2 + BA^2 = GA^2 + 2GF.GA + GF^2$, by addition; and since $FA^2 + BA^2 = FB^2$, we have $FB^2 = GA^2 + 2GF.GA + GF^2$;

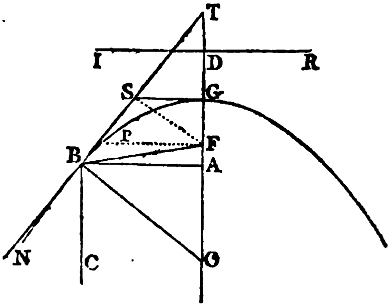
and by extracting the roots, $FB = GA + GF = GA + GD = DA$.

If the point (b) in the curve is above the focus, then $Fa = GF - Ga$, and $Fb = Da$.

Corol. This Theorem affords a ready method of describing the parabola by points, thus: Since the distance of the curve at the extremity of any ordinate from the focus is equal to the distance of that ordinate from the point D, if a number of lines *ab*, FP, AB, &c. are drawn parallel to DI (at right angles to DA) and the distances D*a*, DF, DA, &c. set off from the focus F to meet those lines respectively, the points of concourse will be those through which the curve must be drawn.

The line IDR is called the *directrix* of the parabola.

297. *To draw a tangent to the parabola at a given point* B *in the curve.*



From B draw BF to the focus, and BC parallel to the axis; let BO bisect the angle CBF; then if the angle OBT be made a right one, BT will be the tangent required.

This construction results from considering the parabola as an ellipse whose transverse axe is infinite in length (268). For a tangent to the ellipse at any point is perpendicular to the line which bisects the angle formed by the two lines drawn from the foci to meet the curve at that point (272): if therefore the axis is infinite, one of the foci will be at an infinite distance, and the line drawn from that focus must, in *that case*, be parallel to the axis.

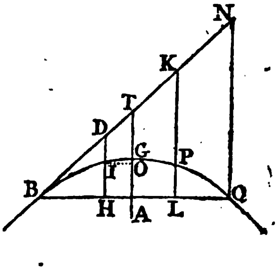
Corol. 1. Hence, because BO is perpendicular to the curve at B, and the angle FBO $=$ OBC, if the concavity of the parabola were a polished surface, all rays of light (as CB, &c.) falling on that surface parallel to the axis, would be reflected to the focus F. (253.)

Corol. 2. From this construction and the preceding theorem, it appears that the *subtangent* TA is bisected at the vertex G, that is, $GA = GT$, (BA being an ordinate to the axis). For the angles NBO, TBO being right ones, and the angle $CBO = FBO$, therefore the angle $NBC = TBF$, but the angle $NBC = BTF$, therefore the angles FBT, FTB are equal, and consequently $FT = FB$. But $FB = DA$ (IDR being the directrix), from this take GD, and from FT take its equal GF, and there remains $GT = GA$.

Corol. 3. Hence also, the distance AO is always = FP or half the parameter. For since $FT = FB$, and the angle TBO a right one, F will be the center of a circle passing through T, B, O, and therefore $FO = FT = FB$: but $FB = AF + 2FG$ (or FD) $= AF + AO$, and consequently $AO = 2FG = FP$.

Corol. 4. And the tangent GS is a mean proportional between GF and GA: for BT is bisected by GS, and the angles FST, SGT being right ones, SG is a mean proportional between GF and GT (Geom. 164) or between GF and GA. And FS is a mean proportional between FG and FT.

298. *If* BN *be a tangent at* B, *and the lines* HD, LK, QN, &c. *are parallel to the axis* GA, *those lines will be divided in* I, G, P, &c. *in the same proportion as the double ordinate* BQ *is divided in* H, A, L, &c.



That is, $ID : IH :: HB : HQ$, &c.
$PK : PL :: LB : LQ$, &c.

Draw the ordinate IO, and let the parameter be denoted by p:

Then because $p \times AG = BA^2$, (268) it will be $p : 2BA$ (or BQ) $:: BA : 2AG$ or AT:

But by sim. triangles, $BH : HD :: BA : AT$, whence by equality $p : BH :: BQ : HD$;

Moreover, $p \times GO = IO^2$ or HA^2, (268)
and $p \times GA = BA^2$,
whence $p\,(GA - GO) = BA^2 - HA^2$, by subtraction,
or $p \times IH \;\; = BA^2 - HA^2$;
therefore, $p : BA + HA :: BA - HA : IH$,
that is, $p : HQ :: BH : IH$;
or alternately, $p : BH :: HQ : IH$;
whence by equality, $BQ : HD :: HQ : IH$,
or alternately, $BQ : HQ :: HD : IH$;
and by division, $BQ - HQ : HQ :: HD - IH : IH$;
That is, $BH \;\; : HQ :: ID : IH$.

Corol. Hence the external lines ID, GT, PK, &c. will have the same ratio as the squares of the corresponding tangents BD, BT, BK, &c.

That is $ID : PK :: BD^2 : BK^2$, &c.

For $ID : IH :: BH : HQ$,
and $ID : IH :: BH^2 : BH.HQ$, by equality,
or $ID : BH^2 :: IH : BH.HQ$, by alternation:

But $\frac{BH.HQ}{p} = IH$, therefore $ID : BH^2 :: \frac{BH.HQ}{p} : BH.HQ$.

In like manner $PK : LB^2 :: \frac{LB.LQ}{p} : LB.LQ$;

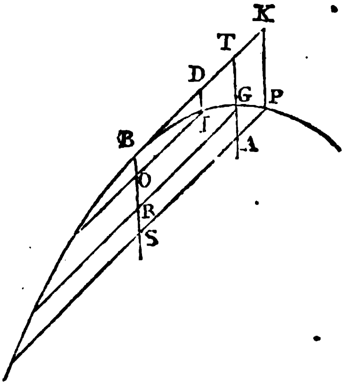
And therefore $ID : PK :: BH^2 : LB^2 :: BD^2 : BK^2$, by sim. triangles.

299. *If* BK *be a tangent to the parabola at* B, *then* BS *parallel to the axis* GA, *is a diameter, and* OI, RG, SP, &c. *parallel to the tangent* BK, *are ordinates to that diameter.*

And the abscissas BO, BR, BS, &c. *have the same ratio as the squares of their corresponding ordinates* OI, RG. SP, &c.

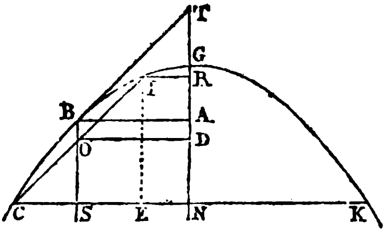
Let ID, GT, PK, &c. be parallel to the axis GA or to the diameter BS. Then BOID, BRGT, &c. being parallelograms, the opposite sides will be respectively equal:

And (298 corol.)
$ID : PK :: BD^2 : BK^2$, &c.
that is
$BO : BS :: OI^2 : SP^2$, &c.



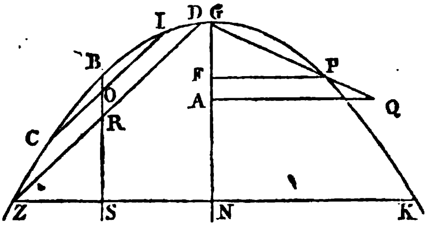
300. *Any diameter* (BS) *bisects all its double ordinates* (IC, &c.) *or lines parallel to the tangent* (BT) *at the vertex* (B) *of that diameter.*

Let IR, BA, CK be ordinates to the axis, and draw IE perpendicular and OD parallel to CN: also suppose $p =$ the parameter.



Then (298) $p \times EI = CE^2 - EN^2 = EK \times EC$,
that is $p : EK :: EC : EI$:
And by sim. triangles $BA : AT$ or $2GA :: EC : EI$,
whence by equality $p : EK :: BA : 2GA$,
or alternately $p : BA :: EK : 2GA$:
But (268) $p : BA :: 2BA : 2GA$,
therefore by equality $EK : 2BA :: 2GA : 2GA$;
consequently EK or $IR + CN = 2BA$. That is, the ordinate BA is an arithmetical mean between the ordinates IR and CN: but $OD = BA$, whence it follows that RN and IC are both bisected by OD.

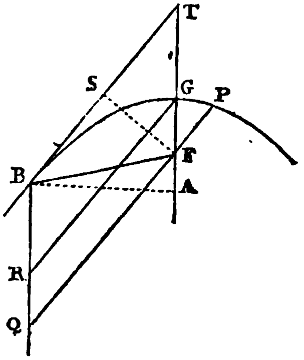
Corol. Hence when the curve of a parabola is given, the axis and focus are determined by the following construction:



Draw any two parallel lines or ordinates IC, DZ terminated by the curve, and bisect them in O, R with the diameter BS; then, at right angles to BS draw ZK which bisect in N with the perpendicular NG, which will be the *axis.*

To find the *focus,* let AQ be parallel to NK and = 2AG, draw QG, and the point P where it intersects the curve, will be the extremity of the parameter of the axis: for by sim. triangles FP = 2FG, therefore F is the *focus.*

301. *Let* GA *be the axis, and* F *the focus; then* (295) *the parameter* (p) *of the axis, is equal to* 4FG *or four times the distance of the focus from the vertex* G. *In like manner, if* B *be the vertex of any other diameter, its parameter* (P) *will be* 4 *times the line drawn from the focus to that vertex. That is,* $P = 4FB$.



Draw GR parallel to the tangent BT meeting the diameter BR in R, also let BA be an ordinate to the axis, and make FS perpendicular to BT.

Then (297. corol. 2) GA = GT = BR, therefore the abscissas GA, BR to the ordinates BA, GR are equal:

And (268) $GA = \frac{AB^2}{p}$; also BR (or GA) $= \frac{GR^2}{p}$ (by the definition);

whence $p : P :: AB^2 : GR^2$ or BT^2:

But $FS^2 : FT^2 :: AB^2 : BT^2$ (by sim. triangles),

therefore $p : P :: FS^2 : FT^2$: but $FS^2 = FG.FT$ (297. corol. 4),

consequently $p : P :: FG.FT : FT^2$,

whence $p : P :: FG : FT$ or FB;

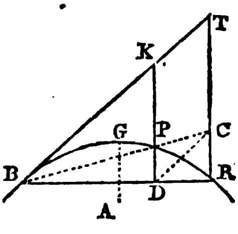
but $p = 4FG$, and therefore $P = 4FB$.

Corol. Let PFQ be parallel to GR. Then because $FB = FT = BQ$, we have $P = 4BQ$ the parameter of the diameter BR; therefore (by the definition) the parameter is the double ordinate drawn through Q, and consequently $2BQ = QP$ the semiparameter. Whence also it appears, that the parameters of all the diameters of a parabola pass through the focus.

And it may be observed in general, that the properties which have been demonstrated respecting the axis, its abscissas, and ordinates, extend to any other diameter, its abscissas and ordinates.

302. *Let* BR *be any right line terminated by the curve, and* BT *a tangent at* B; *then if* KD *be a line parallel to the axis* GA, *it will be divided by the curve at* P *in the same ratio as* BR *is divided in* D:

That is, $PD : KP :: DR : BD$.



Draw RT parallel to DK:

Then (299) $KP : TR :: BK^2 : BT^2$.

And $BD^2 : BR^2 :: BK^2 : BT^2$. (by sim. triang.)

therefore by equality

$KP : TR :: BD^2 : BR^2$, whence $KP.BR^2 = TR.BD^2$;

Again, by sim. triang. $KD : TR :: BD : BR$,

and $KD : TR :: BD^2 : BD.BR$, or $KD\ (BD.BR) = TR.BD^2$,

therefore $KP.BR^2 = KD\ (BD.BR)$, or $KP.BR = KD.BD$;

that is $KD : KP :: BR : BD$,

And $PD : KP :: DR : BD$, by division.

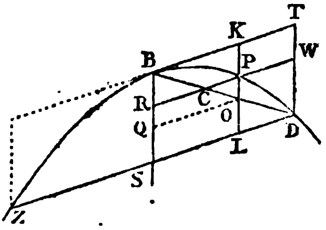
Corol. If BC be drawn through the point of intersection P, then DC is parallel to the tangent BT. For the triangles BDP, BRC being similar,

we have BP : PC :: BD : DR,
but KP : PD :: BD : DR,

whence, by equality, BP : KP :: PC : PD, therefore (Geom. 94, corol. 1) the triangles BKP, DCP are equiangular, and DC parallel to BK.

303. *If* BS *be any diameter,* BT *a tangent at* B, *and* ZT *a parallelogram described about the parabola; then if* KL *be a line parallel to* BS, *and* BD *joined,* KO *will be a mean proportional between* KP *and* KL:

That is, KP : KO :: KO : KL.



Draw RPW and QO parallel to BT or SD:

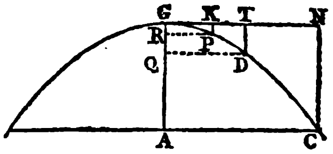
Then (299) BR : BS :: RP^2 or QO^2 : SD^2 (by sim. triang.) :: BQ^2 : BS^2.

And BR.BS : BS^2 :: BQ^2 : BS^2, therefore BR.BS $=BQ^2$,
or BR : BQ :: BQ : BS,
That is, KP : KO :: KO : KO.

Corol. Since BR : BQ :: BQ : BS, therefore BQ is a mean proportional between BR and BS: but by sim. triangles, RC, QO (or RP) and SD (or RW) are in the same proportion as BR, BQ, and BS, and consequently RW is divided in C and P so that RC, RP, and RW are also in continued proportion, or RC : RP :: RP : RW.

304. *The area of a parabola is* $\frac{2}{3}$ *of its circumscribing parallelogram:*

That is, the space AGPC $=\frac{2}{3}$AGNC.



Conceive the surface GPDCN to be composed, or made up of an indefinite number of indefinitely small threads or lines KP, TD, &c. parallel to NC or the axis GA, the longest being, NC, and the shortest at $G = o$:

Then (268) $GA : AC^2 :: GR : RP^2$ or GK^2,

$$\text{therefore } GR \text{ or } KP = \frac{GA}{AC^2} \times GK^2:$$

$$\text{In like manner } TD = \frac{GA}{AC^2} \times GT \; ;$$

$$\&c. \qquad \&c.$$

Hence $\left(\frac{GA}{AC^2}\right.$ being a constant quantity$\left.\right)$ the sum of all the lines KP, TD, &c. will be $\frac{GA}{AC^2} \times (o^2 + GK^2 + GT^2 + \ldots GN^2)$:

Now if GN is supposed to be divided into an indefinite number of indefinitely small and equal parts, these parts will form an arithmetical progression whose least term is o, and greatest GN; and if n is put to denote the number of terms, the sum of their squares, or the sum of the infinite progression will be $\frac{n^3}{3}$ (179); the first term being o^2, and last n^2.*

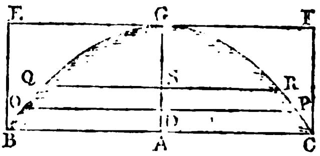
---

* Should the student have any doubts respecting this result from art. 179, the following process will show the truth of the conclusion. Let n or GN be the perpendicular height of a pyramid having a square base whose side is also = GN, and conceive the pyramid to be composed of an infinite number of indefinitely thin square laminæ laid one upon another, the greatest or the base being GN^2, and the least at the vertex $= o^2$; then it is evident the content of the pyramid will be the sum of all the laminæ or series of squares from o^2 to GN^2; but the content of the pyramid is $= GN^2 \times \frac{1}{3}GN = \frac{1}{3}GN^3$, the sum of the series, as above, according to the *Arithmetic of Infinites.*

The summation of such series however, is properly the business of Fluxions, which affords a general method: but the expressions in the article refered to, will answer the purpose in some of the most simple cases.

But GN is supposed to be divided into n equal parts, consequently $n = GN$; and $\frac{1}{3}n^3 = \frac{1}{3}GN^3$; and therefore $\frac{GA}{AC^2} \times (o^2 + GK^2 + GT^2 + \ldots\ldots GN^2)$ becomes $\frac{GA}{AC^2} \times \frac{1}{3}GN^3 = \frac{1}{3}GA \times GN$ (because $AC = GN$) the content of the space GPCN; therefore $AGPC = \frac{2}{3}GA \times GN$, or $\frac{2}{3}$ of the parallelogram AN.

305. *The content of a paraboloid or solid generated by the revolution of a parabola* (BGC) *about its axis* (GA), *is half its circumscribing cylinder.*



Suppose the axis of the parabola divided into an infinite, or indefinite number of equal parts, and conceive the paraboloid to be composed of the like or corresponding number of circular sections whose diameters are BC, OP, QR, &c. the diameter of the greatest section being BC, and that of the least section (at G) $= o$:

Then the number of sections, or parts into which GA is supposed to be divided, form an arithmetical progression whose first term $= o$, last term $= GA$, and number of terms also $= GA$; and the sum of such a series $= (o + GA) \times \frac{1}{2}GA$, or $\frac{1}{2}GA^2$.

By Art. 268, we have $GA : BC^2 :: GD : \frac{BC^2}{GA} \times GD = OP^2$,

And $GA : BC^2 :: GS : \frac{BC^2}{GA} \times GS = QR^2$;

&c. &c. &c.

Let $c = .7854$

Then cBC^2 or $\frac{cBC^2}{GA} \times GA =$ the circular section whose diam. is BC;

cOP^2 or $\frac{cBC^2}{GA} \times GD =$ section whose diam. is OP;

$$cQR^2 \text{ or } \frac{cBC^2}{GA} \times GS = \text{that having the diam. } QR:$$

&c. &c.

consequently $\frac{cBC^2}{GA} \times (GA + GD + GS + \ldots\ldots o)$ will be the sum of all the circular sections, or content of the paraboloid: but the sum $(GA + GD + GS + \&c. \ldots\ldots o) = \frac{1}{2}GA^2$, hence the expression becomes $\frac{cBC^2}{GA} \times \frac{1}{2}GA^2$ or $cBC^2 \times \frac{1}{2}GA$, which is half the content of the circumscribing cylinder BEFC.

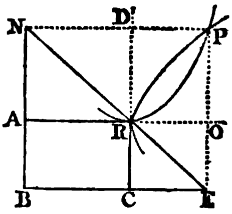
OF THE CONSTRUCTION OF

CUBIC AND BIQUADRATIC EQUATIONS.

306. We have given the construction of quadratic equations by means of right lines and the circle (233): but either of the conic sections might be substituted for the latter, because their equations are also of two dimensions: the circle however, is prefered on account of the simplicity of its description. A right line can intersect a conic section in two points only, which determine the two roots of a quadratic. But one conic section may cut another in as many points as a cubic, or a biquadratic equation has roots; hence it appears that such equations can be constructed by means of the conic sections, or their roots determined by the intersections of *loci* of two dimensions.

307. *To construct a simple cubic equation* $x^3 = a^2 b$, *or to find two mean proportionals between two right lines denoted by* a *and* b.

We shall take the example Art. 254, where it is required to find two mean proportionals between the lines RC and BC or RA.

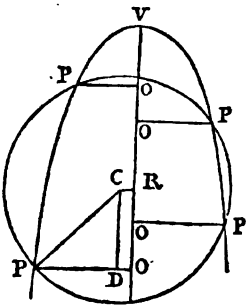


Let the angle ARC be a right one: produce CR and AR, and on the axis RD describe a semiparabola RPD having its parameter =RC, and on the axis RO another RPO whose parameter = RA: then the ordinates PD, PO drawn from the intersection P to the axes, will be the mean proportionals required.

For (268) RD × *param.* RC = PD^2, and RO × *param.* RA = PO^2, therefore PD is a mean proportional between RC and RD or PO; and PO is a mean proportional between RO or PD and RA.

This problem is usually constructed by means of the circle and one of the conic sections: the preceding method however, is more simple of explication.

308. *To construct a Biquadratic.* Let the circle whose center is C intersect the parabola PVP in the points P, P, P, P; draw the ordinates PO, PO, PO, PO to the axis VO; also make CD parallel and CR perpendicular to VO, and draw CP.



Put VO = x, OP = y, VR = a, CR = b, CP = r, and the parameter of the parabola = p. Then $px = y^2$, whence $x = \frac{y^2}{p}$. Also PD = PO — DO = $y - b$, and CD = VO — VR = $x - a$: but CP^2 = PD^2 + CD^2, that is $(x-a)^2 + (y-b)^2 = x^2 - 2ax + a^2 + y^2 + b^2 = r^2$, and substituting $\frac{y^2}{p}$ for x, we get

$$\left.\begin{matrix} y^4 - 2pa \\ \times\ p^2 \end{matrix}\right\} y^2 - 2p^2by + (a^2 + b^2 - r^2)p^2 = o,$$ which by varying the values of the coefficients, may be made to coincide with any proposed biquadratic equation that wants the second term; and then the ordinates on the axis from the points of intersection P, P, P, P, will be the roots of that equation.

For example, suppose the proposed equation to be $y^4 - my^2 + ny - c = o$.

Let a parabola PVP be described whose parameter $= 1 = p$;

then $\left.\begin{array}{l}-2pa\\+\ p^2\end{array}\right\} = 1 - 2a = m$, whence $a = \frac{m+1}{2}$:

$-2p^2b = -2b = n$, or $b = -\frac{n}{2}$:

$(a^2 + b^2 - r^2)p^2 = a^2 + b^2 - r^2 = -c$, whence $r = \sqrt{(a^2 + b^2 + c)}$:

Now in the axis take $VR = a = \frac{m+1}{2}$, and make RC perpendicular to VR and $= -\frac{n}{2} = b$, then about the center C, with the radius $\sqrt{(a^2 + b^2 + c)}$ describe a circle, and the ordinates to the axis from the points of intersection P, P, P, P, will be the four roots of the equation.

When RC represents a negative quantity, the ordinates on that side of the axis are the negative roots, and the contrary.

Corol. 1. If the circle cut the parabola in two points only, the equation has but two real roots, the others being imaginary: and if it touch the parabola, two roots must be equal, because two of the ordinates may be said to coincide.

Corol. 2. Should the circle pass through the vertex V, then $CP^2 = CR^2 + VR^2$, that is, $r^2 = b^2 + a^2$, and the last term of the biquadratic will vanish, if therefore the remainder be divided by y, the result is

$$y^3 \left.\begin{array}{l}-2pa\\+\ p^2\end{array}\right\} y - 2p^2b = o,$$

which may be made to coincide with any *cubic equation* wanting the second term, and the ordinates will be reduced to three for its roots.

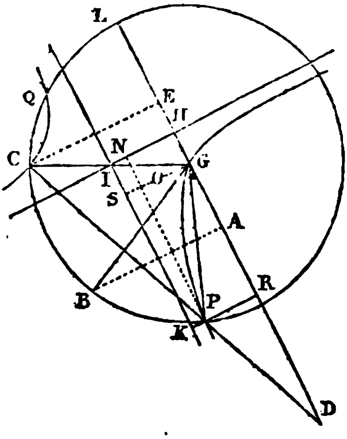
This method of constructing biquadratic and cubic equations which want the second term, is that of Descartes. But the constructions may be made general by refering the lines denoting the roots to a diameter of the parabola that is not the *axis*, as may be seen in Baker's *Geometrical Key*, l'Hospital's *Conics*, Maclaurin's *Algeb.* &c.

But the same thing may be effected by making use of either of the other conic sections, instead of the parabola, which is usually assumed because its equation consists of two terms only: the ellipse however, is more easy of description.

309. *An angle may be trisected by the construction of a cubic equation.* Thus if s and c denote the *sine* and *cosine* of an arc, the *radius* being 1: then (246) $3sc^2 - s^3$ is the *sine* of three times that arc; but $c^2 = 1 - s^2$ which being substituted for c^2, and we have the cubic equation $3s - 4s^3 = A$, putting $A =$ the *sine* of the angle to be trisected, and consequently the value of s will be the *sine* of $\frac{1}{3}$ of the proposed angle.

The following construction of the problem is by means of the *circle* and *hyperbola*.

Let BGA be the given angle. About G with the radius GB describe a circle; make BA perpendicular to GA, and in AG produced take GH $= \frac{1}{2}$GA; draw HI parallel and equal to $\frac{1}{2}$AB, and let IK be parallel to HA; then between the asymptotes IH, IK describe an hyperbola to pass through the point G, and it will cut the circle in P so that the angle PGA $= \frac{1}{3}$ of the angle BGA.



Through I draw the radius GC, and let CE be perpendicular to GE, also make GS and KPR parallel to IH, and PN to KI.

Then since GH and HI are the halves of GA and AB, the triangles GHI, GAB are similar, and because GC = GB, the triangles GCE, GBA are similar and equal.

And because the parallelograms NK, HS are equal (292) if the common parallelogram NS be subtracted from each, and the parallelogram OR added, the parallelogram SR = parallelogram NR, that is PR × RH = KR × RG,

or PR : RG :: KR : RH,
and :: 2KR : 2RH, (by doubling)
or :: CE : 2RH, (because 2KR = CE);

But because HE = HG, therefore 2RH = ER + RG,
hence we have PR : RG :: CE : ER + RG,
or CE : PR :: ER + RG : RG.

But CE : PR :: ED : RD, by sim. triangles;
whence ED : RD :: ER + RG : RG, by equality,
or ED − RD (ER) : RD :: ER : RG, by division;
therefore RD = RG, and consequently PD = PG:

Now GP = GC, and therefore the angle GCP = GPC; but the external angle CPG = PDG + PGD, or CPG = twice PGD: in like manner, the external angle CGE = GCD + GDC, or CGE = triple PGD, *that is* BGA (= CGE) = *triple* PGD.

If the opposite hyperbola be described, it will trisect the supplemental angle BGL, that is, the arc QL is $\frac{1}{3}$ of the arc BCL.

Since all parallelograms inscribed between the asymptotes and curve are equal, the semitransverse or semiconjugate axis of the hyperbola, will be the diagonal of a square whose side is $\sqrt{(GS . GH)}$.

310. OF MECHANICS.

DEFINITIONS.

1. MECHANICS is the science which treats of the motions, velocities, forces, and in general of the actions and effects of moving bodies upon one another. It comprehends *Statics*, on the weight and equilibrium of solid bodies. *Dynamics*, the science of moving powers. *Hydrostatics*, of the gravity, and pressure of fluids. *Hydraulics*, treating of the motion of water, and other fluids, the construction of water-works, &c. &c.

2. *Motion* is a constant and successive change of place. If the body moves equably or passes over equal spaces in equal times, it is called uniform motion. If it increases, or decreases, it is called accelerated, or retarded motion. The motion is also said to be absolute, or relative, according as the body moved is compared with another body at rest, or in motion.

3. *Velocity* or *celerity*, is the quickness or slowness of motion, or the rate at which a body moves. Thus if a body passes uniformly over a space of two feet in half a second of time, it is said to have a velocity of 4 feet per second, or move at the rate of 4 feet in a second.

4. *Quantity of motion* or *momentum*, is the power or force of bodies in motion. This is proportional to the weight or quantity of matter moved drawn into its velocity.

5. *Force* is a power exerted on a body to put it in motion. If it act instantaneously, it is called *impulse* or percussion. If constantly, it is a permanent force like pressure or the force of gravity.

6. *Forces* are also distinguished into *motive*, and *accelerative* or retardive. The motive or moving force relates to the quantity of matter moved as well as the velocity communicated, and is proportional to the momentum or quantity of motion produced in a given time.

7. An accelerating or retarding force is generally understood to be that which affects the celerity only, and therefore it is proportional to the velocity generated in a given time, or to the motive force directly, and the mass or body moved inversely.

Thus if the body or mass B be urged by the moving force F, then $\frac{F}{B}$ will denote the accelerating force; for the magnitude or value of the fraction $\frac{F}{B}$ increases directly as F is increased, but diminishes as B is augmented.

Gravity or the power of gravitation is an accelerating force; for the velocity of a body falling by its own weight, or projected vertically, is continually augmented in the former case, but diminished in the latter, till all its motion in that direction is lost.

8. *Vis inertiæ*, is the innate force of a body by which it resists any endeavour to change its state; this is always proportional to the quantity of matter in the body. Thus if two bodies of the same kind are floating on water, the less or lighter body is more easily moved than the greater, and therefore its vis inertiæ is less.

9. An *elastic* body is that, the position of whose parts being changed by the action of a force, either recovers, or has a tendency to recover its former figure. Thus the strings of a violin are elastic. And a tennis ball rebounds by the force of its elasticity or the force exerted by its parts in recovering their position before impact. Bodies not having this property are called *non-elastic*.

10. *Gravity* or Weight, is that force by which a body endeavours to fall downwards. Absolute gravity is when the body is in free space; and relative gravity when it is immersed in a fluid.

11. *Specific gravity*, is the proportion of the weights of different bodies of equal magnitudes. Thus if a globe of wood or other matter of 4 inches diameter will sink by its own weight just 2 inches in water, its specific gravity to that of the water is as 1 to 2.

12. *Density* is also the proportion of the quantity of matter in any body to the quantity in another body of the same magnitude. Thus if a body of any size weigh 6 pounds, and another of equal bulk weigh 4 pounds, the density of the former to that of the latter is as 3 to 2.

13. *Friction* is a retarding force in machines, arising from the parts rubbing against one another.

311. AXIOMS.

1. Every body perseveres in its state of rest, or uniform motion in a right line, unless it be compelled to change that state by some external force.

2. The alteration or change of motion is always proportional to the force applied, and is made in the direction of that right line in which it acts.

3. Action and reaction are equal and in contrary directions.

OF THE GENERAL LAWS OF MOTION, &c.

312. THE *quantities of matter in all bodies, are in the compound ratio of their magnitudes and densities.*

The quantity of matter in a body may be denoted by its weight; therefore

if $w =$ the body or its weight;

$m =$ its magnitude in cubic feet, or any other known measure;

$d =$ its density:

then w is as md, or w is always directly proportional to $m \times d$.

Let $w : m \times d :: a : b$; and suppose the density to be doubled, then the weight must also be double, the magnitude remaining the same,

hence $2w : m \times 2d :: a : b$:

Again, if the magnitude be tripled, it is manifest the weight will also be increased 3 times, and so on:

consequently $2w \times 3 : 3m \times 2d :: a : b$,

That is, the weight or quantity of matter $6w$ is directly proportional to the magnitude $3m$ multiplied by the density $2d$.

Corol. If the magnitude be given, the weight is as the density. And when the density is given, the weight will be as the magnitude.

313. *The momentum or quantity of motion generated by an impulse or momentary force, is as the force that generates it.*

For a double force will manifestly generate a double quantity of motion or momentum; a triple force a triple momentum, and so on. That is, the motion impressed is directly as the percussive or motive force which produces it.

314. *The spaces described in uniform motions, are in the compound ratio of the velocities and the times of their description.*

Thus if the velocity be v feet per second, and the time $= t$ seconds, then the space described in the time t will be $v \times t$ feet: that is, the space is directly as vt. And if $s =$ the space in feet, then $s = vt$.

Corol. 1. Hence if the time be the same, the space described will be as the velocity: but when the velocity is the same, it will vary as the time.

Corol. 2. Since $t = \frac{s}{v}$, and $v = \frac{s}{t}$: therefore in uniform motions, the time is as the space directly, and velocity reciprocally. And the velocity is as the space directly and time reciprocally.

315. *Let m denote the momentum or quantity of motion in a moving body, w its weight or quantity of matter, and v its velocity; then if they are supposed to be variable, m will vary as w × v. That is, the momentum will be in the compound ratio of the mass and velocity.*

If a body be put in motion with any initial velocity by a momentary force, it is manifest that double that force will be necessary to communicate a double velocity, and a triple velocity will require a triple force, and so on: now the momentum being as the generating force (313) it follows, that in the same body, the momentum is as its velocity; but the momentum also increases as the quantity of matter increases, consequently in all bodies it must be as the mass and velocity jointly: or m is directly proportional to wv.

Corol. 1. Hence the ratio of the momenta of two bodies in motion, is compounded of the ratios of their masses and velocities. For let the momentum, weight or mass, and velocity of a body be denoted by M, W, and V, respectively,

$$\text{then } m : w \times v :: M : W \times V$$

$$\text{That is } \frac{m}{M} = \frac{w}{W} \times \frac{v}{V}.$$

Corol. 2. Since $\frac{m}{M} = \frac{w}{W} \times \frac{v}{V}$, we have $\frac{v}{V} = \frac{m}{M} \times \frac{W}{w}$, that is, the ratio of the velocities is compounded of the direct ratio of the momenta, and the reciprocal ratio of the weights or quantities of matter.

SCHOLIUM. To exemplify this proposition in numbers, suppose two cannon shot, one 9*lb.* the other 36*lb.* to strike an obstacle with the respective velocities of 1000 and 800 feet per second; then their momenta or the forces with which they meet the obstacle will be as 9×1000 and 36×800, or as 5 to 16. In this manner the forces of impact or percussion are compared one with another. But it may be observed that such forces cannot be compared with the force of pressure or weight, or bodies at rest, no more than a rectangle can be compared with the line by which it is generated.

316. *If a quiescent body be urged by an uniformly accelerating force during a given time, the velocity generated at the end of that time will be in the compound ratio of the force and time of acting.*

Let t denote the time, and f the constant force; and conceive the time to be divided into innumerable equal particles; then the first impulse will manifestly generate in the body a velocity proportional to the acting force f, which velocity may be considered uniform during the first particle of time, we can therefore denote this velocity by f because it is proportional to that force; now while the body is moving with the velocity f, it receives another impulse equal to the former, which must generate an equal velocity, the body therefore in the second particle of time will move with a celerity proportional to $f + f$ or $2 \times f$; in like manner $3 \times f$ will denote the velocity during the 3*d.* particle of time, and so on; consequently the last velocity or that during the tth. or ultimate particle of time will be represented by $t \times f$.

And in uniformly retarded motions, the diminished velocity will also be in the compound ratio of the retarding force and time.

Corol. 1. Therefore in uniformly accelerated, or retarded motions, the increments, or decrements of velocity, are equal in equal times, because f, $2f$, $3f$, &c. form an arithmetical progression. And hence we can determine the relation between the time and space described; for it is evident that the space described in the time t with the successive velocities f, $2f$, $3f$, &c. would also be described in the same time with an uniform velocity which is a mean between all the velocities or terms of that series: now the greatest velocity or greatest term of the progression is tf; and as the particles of time are supposed to be indefinitely small, the least term may be taken $= o$; and the number of terms being $= t$, we have $o + f + 2f + 3f + \ldots\ldots tf = (o + tf) \times \frac{1}{2}t$, or $\frac{1}{2}t^2f$ the sum of all the terms or celerities, which being divided by t their number, gives $\frac{1}{2}tf$ the mean velocity, equal to half the greatest (tf); hence it appears, that if the body moved uniformly with half its greatest celerity, it would describe the same space in the same time. Now the space being in the compound ratio of the velocity and time (314) it will therefore be as $\frac{1}{2}tf \times t$ or $\frac{1}{2}t^2f$, that is, as t^2 the square of the time, the force f and body remaining the same. And because the velocities generated, or destroyed, are as the times of description, the space will also be as the square of the velocity. If the body varies, the velocity (with the same force) is inversely as the mass or weight, in which case the space described will be directly as the force and square of the time, and reciprocally as the mass.

Corol. 2. Since in the same body, the momentum is as its velocity, therefore the momentum generated or destroyed by an uniformly accelerating or retarding force acting for any time, is also in the compound ratio of the force and time of acting.

SCHOLIUM.

Let w = the weight or mass or quantity of matter in a body,
f = the force constantly acting on it,
t = the time of its acting,
v = the velocity generated in that time,
s = the space described,
m = the momentum at the end of the time t:

Then $\propto$ being the symbol denoting general proportion, we have, from the two last articles, the following relations in uniformly accelerated motions,

$$m \propto wv \propto tf.$$
$$v \propto tf.$$
$$s \propto vt.$$
$$s \propto \frac{t^2 f}{w}.$$
$$\left.\begin{array}{l} s \propto t^2 \\ s \propto v^2 \end{array}\right\} \text{when the force and mass are proportional.}$$

And from these proportions or relations, other comparisons are readily derived. Thus, since equimultiples or submultiples of quantities have the same ratio as the quantities themselves, if (for example) we divide $wv \propto tf$ by w the result is $v \propto \frac{tf}{w}$, that is, the velocity generated or destroyed in any given time, is directly as the force and time, and inversely as its weight or mass when the latter is not given.

Since $s \propto t^2 \times \frac{f}{w}$ is the same as $s \propto \frac{t^2 f}{w}$, if the force ($f$) and mass or weight ($w$) are proportional, then omitting $\frac{f}{w}$, we have $s \propto t^2$: for, by the nature of fractions, when s is as $t^2 \times \frac{f}{w}$, and f as w, s will be as t^2, or $s \propto t^2$, as above. This takes place in bodies acted on by gravity, where the force is proportional to the weight or quantity of matter. But (310 def. 7) if $\frac{f}{w}$ (the accelerating force) $= F$, then $s \propto t^2 F$, whence $t \propto \sqrt{\frac{s}{F}}$. And because $s \propto vt$, we have $t \propto \frac{s}{v}$, therefore by substitution $\frac{s}{v} \propto \sqrt{\frac{s}{F}}$, and $s \propto \frac{v^2}{F}$, whence $v \propto \sqrt{sF}$.

Hence we shall have

$$v \propto \sqrt{sF} \propto Ft.$$
$$t \propto \sqrt{\frac{s}{F}} \propto \frac{v}{F}.$$
$$s \propto t^2 F \propto \frac{v^2}{F}.$$

And given quantities are also to be left out. Thus s varies as vt, or $s \propto vt$; now if v the velocity is given, then $s \propto t$, or the space will vary as the time.

317. To compare the velocities, &c. of two bodies, let W denote any other weight or mass, and F, T, V, S, M, the acting force, time, &c. as above;

Then

$$v : \frac{tf}{w} :: V : \frac{TF}{W}, \text{ whence } \begin{cases} \dfrac{v}{V} = \dfrac{t}{T} \times \dfrac{f}{F} \times \dfrac{W}{w} \\ \dfrac{t}{T} = \dfrac{v}{V} \times \dfrac{F}{f} \times \dfrac{w}{W}. \end{cases}$$

$$s : vt :: S : VT \ldots\ldots\ldots \frac{s}{S} = \frac{v}{V} \times \frac{t}{T}.$$

$$s : \frac{t^2 f}{w} :: S : \frac{T^2 F}{W} \ldots\ldots\ldots \frac{s}{S} = \frac{t^2}{T^2} \times \frac{f}{F} \times \frac{W}{w}.$$

&c. &c.

But numeral results are obtained from quantities denoted by numbers. We shall subjoin an example or two. Let f = the force of gravity, which may be considered as uniform near the earth's surface. Then since it has been found by experiments that a body descends from rest in a perpendicular direction the space of $16\frac{1}{12}$ *feet* in the first second of time, and because an equal space would be described by the body in the same time if it moved uniformly with half its acquired velocity, (316, corol. 1) its velocity therefore at the end of the first second of time will be $16\frac{1}{12} \times 2$, or $32\frac{1}{6}$ *feet* per second; and the celerity generated or destroyed being as the times of description, we have 1 *sec.* : $32\frac{1}{6}$ :: 2 *sec.* : $32\frac{1}{6} \times 2$, or $64\frac{1}{3}$ *feet* per second the velocity at the end of 2 seconds; and therefore $32\frac{1}{6}t$ feet is the velocity per second which bodies acquire in descending perpendicularly from rest, at the end of t seconds. Also since the spaces described are as the squares of the times, or the squares of the generated celerities, we have

$$t^2 : T^2 :: s : S,$$
$$v^2 : V^2 :: s : S,$$

whence $S = \frac{sT^2}{t^2} = \frac{sV^2}{v^2}.$

$$T = t\sqrt{\frac{S}{s}} = \frac{Vt}{v}.$$

$$V = v\sqrt{\frac{S}{s}} = \frac{Tv}{t}.$$

Suppose it is required to find how far a heavy body would descend by the force of gravity or its own weight in 6 seconds of time, and also its velocity at the end of that time. Then $s = 16\frac{1}{12}$, $t = 1$ *sec.* $T = 12$ *sec.* and $v = 32\frac{1}{6}$,

and we have $S = \frac{sT^2}{t^2} = 16\frac{1}{12} \times 36 = 579$ *feet*, the distance:

And $V = \frac{Tv}{t} = 32\frac{1}{6} \times 6 = 193$ *feet* per second, the celerity.

Admit a shot to be discharged in a perpendicular direction with an initial velocity of 193 *feet* per second; to what height would it ascend, and what time would elapse before it fell to the ground again?

Here $S = \frac{sV^2}{v^2} = \frac{16\frac{1}{12} \times 193^2}{(32\frac{1}{6})^2} = 579$ *feet*, the height:

And $T = \frac{Vt}{v} = \frac{193 \times 1}{32\frac{1}{6}} = 6$ seconds the time of its ascent, therefore 12 seconds is the time required.

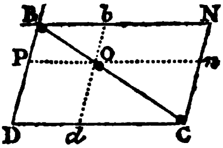
The mass or weight of the body is not considered in these computations, because all bodies would fall equally fast if they were not resisted by the air. The laws of descent therefore suppose that bodies fall in a non-resisting medium. If f and F denote the motive and accelerative forces, respectively, then (310. def. 7) $F \propto \frac{f}{w}$: now if a body descends perpendicular by its own weight or the force of gravity, the weight itself is the motive force, and consequently $\frac{f}{w} = 1$, that is, the accelerative force of gravity F is constant; this is usually expounded by $32\frac{1}{6}$ *feet* the increase of velocity generated by that force in every second of time. Gravity however, strictly speaking, is a variable force, for a body is somewhat heavier near the earth's surface than at any distance above it, because it is more strongly attracted by the earth in the former situation than in the latter.

OF THE

COMPOSITION AND RESOLUTION OF FORCES.

318. WHEN the effects of several forces acting in different directions are reduced to that of a single force acting in one direction only, it it called *composition of forces*. And conversely, the *resolution of forces* consists in finding two or more forces whose joint effect in different directions shall be equivalent to that of a single force in a given direction.

319. *Suppose a body at* B *to be urged in the directions* BD *and* BN *by two forces that would separately cause it to move uniformly along the lines* BD *and* BN *in the time* t; *then if both forces act together, the body, by the compound motion, will describe* BC *the diagonal of the parallelogram* BNCD *in the same time* t.



Conceive BD and BN to be two inflexible lines or wires in contact with the body placed between them at the angular point B; then if the lines begin their motions together and move parallel to themselves in the same plane towards NC and DC, the body will be carried or urged along that plane by the two lines or wires, and constantly move in the angle DBN or *dOn* formed by their intersection, its track therefore must be the diagonal BC; for let *bd* and P*n* be any cotemporary positions of the moving lines, then because BD moves uniformly from the position BD to NC in the same time that BN moves uniformly from BN to DC, their velocities are as the lines BD and BN; and for the same reason, BP and B*b* will denote the velocities when the lines are in the positions *bd* and *Pn*, but the velocities being uniform, the lines BD, BN, and BP . B*b* are therefore proportional, consequently (by sim. triang.) the intersection Q or place of the body will always be in the diagonal, of the parallelogram BNCD, and since it is supposed to be always in contact with the moving lines or wires, its situation at the end of the time t is the angular point C.

And the body will also describe the diagonal BC when urged by uniformly accelerating forces, provided they are similar: For let T and t denote the times of describing BD or BN, and BP or Bb, respectively; then the spaces being as the squares of the times (316, corol. 1.) we have

$$\mathrm{BD} : \mathrm{BP} :: T^2 : t^2,$$
$$\mathrm{BN} : \mathrm{B}b :: T^2 : t^2,$$

whence by equality $\mathrm{BD} : \mathrm{BP} :: \mathrm{BN} : \mathrm{B}b$:

That is, the parallelograms BPOb, BDCN are similar, and therefore the angle dOn or situation of the body is always in the diagonal BC as before. The same thing is also manifest in the case of uniformly retarding forces:

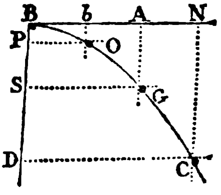
Thus, suppose the motion of BD to be $144\frac{3}{4}$ *feet* in the first second of time, $112\frac{7}{12}$ in the next, $80\frac{5}{12}$ in the third, &c. and that of BN 63 *feet* in the first second, 49 in the next, 35 in the third, &c. then, for example, if *bd*, P*n*; NC, DC, are the positions of the lines at the end of the first, and third seconds of time, respectively, we have $\mathrm{B}b = 144\frac{3}{4}$, $\mathrm{BP} = 63$, $\mathrm{BN} = 337\frac{3}{4}$, and $\mathrm{BD} = 147$ *feet*;

and $\mathrm{BN} : \mathrm{BD} :: \mathrm{B}b : \mathrm{BP}$,
or $337\frac{3}{4} : 147 :: 144\frac{3}{4} : 63$.

Or suppose Bb, BP are described in 2 seconds, then $\mathrm{B}b = 257\frac{1}{3}$, and $\mathrm{BP} = 112$ *feet*; and $337\frac{3}{4} : 147 :: 257\frac{1}{3} : 112$. Therefore the parallelograms BPOb, BDCN, are similar, as before.

Corol. 1. The velocities at the points P, O, b, and consequently the forces in the directions BP, BO, Bb, are as the lines BP, BO, Bb. And the force in the direction BO is equivalent to, or *compounded of*, the two forces in the directions BP and Bb.

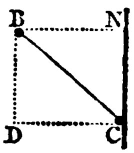
Corol. 2. And since the forces in the directions BP, BO, Bb, may be expounded by those lines, it follows that any single force BO, or BC, can be *resolved into* two other forces acting in different directions by making that force (BO, or BC) the diagonal of a parallelogram.



Corol. 3. If the motion be equable in one direction, and uniformly accelerated in the other, the body will not describe the diagonal of a parallelogram, but the curve of a parabola. For let the motion of the line BN in the direction BD be uniformly accelerated, like that of a body falling from rest towards the earth; and suppose the line BD to move in the direction BN so as to describe each of the equal spaces or lines B*b*, *b*A, A*n*, &c. in the same time *t*; then if PO, *b*O; SG, AG; DC, NC, &c. are the positions of the moving lines at the end of *t*, 2*t*, 3*t*, &c. times, respectively, the intersections O, G, C, &c. will be the corresponding places of the body; and since the lines B*b*, BA, BN, &c. (or their equals PO, SG, DC, &c.) are directly as the times of description, and the distances BP, BS, BD, &c. as the squares of the times (316, corol. 1) it will be

$$BP : PO^2 :: BS : SG^2 :: BD : DC^2, \&c.$$

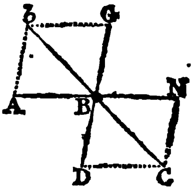
hence the points B, O, G, C, &c. are in the curve of a parabola: And BN is a tangent to the curve at B. (299.)



Corol. 4. Hence also the forces of oblique and direct impact may be compared: Thus, suppose a body to be urged from B in the direction BC by a force denoted by the line BC, then if that force be resolved into two other forces BD and DC (or BN) the former parallel and the latter perpendicular to an obstacle NC, the line DC will represent the force exerted by the body against the obstacle; that is, as BC : DC :: *force in the direction* BC : *force in direction* DC: but DC (or BN) is the *sine* of the incident angle BCN to the *radius* BC; therefore we shall have; As *radius* : *sine of obliquity of the force* :: *force of direct impact* : *force of oblique impact.*

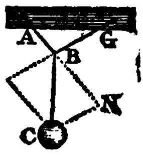
For example, suppose a 48*lb.* shot when moving with a velocity of 1000 *feet* per second should strike an object (NC) in an angle of 50° (NCB), then 48×1000 will denote its momentum, and *rad.* : *sine* 50° :: 48×1000 : 36770; therefore its force against the obstacle will be less than it would be in a perpendicular direction in the proportion of 36770 to 48000.

320. *If three forces of the same kind* A, G, C, *act together in the same plane against the body* B *in the directions* AB, GB, CB, *and thereby keep it in equilibrio, those forces will be proportional to the sides of a triangle* BDC (*or* BNC) *which are drawn parallel to the directions* AB, GB, *and* CB.



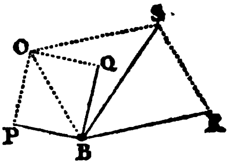
This is manifest from the last proposition corol. 1. for since the force (*b*B) in the direction BC is equivalent to, or compounded of, the two forces in the directions AB or BN and GB or BD, if the former (BC) be exerted in a contrary direction (CB) the effects of the other two will be destroyed, and the body must remain quiescent; the three forces therefore are as BN, NC (or BD), and GB, the sides of the triangle BNC or its equal BA*b*.

Corol. 1. And because three lines perpendicular to the sides of a triangle will form another similar triangle, the three forces will also be proportional to the sides of that similar triangle.



Corol. 2. Hence if the force in the direction BC be a weight C suspended by three strings or cords AB, GB, BC, the tensions of the cords or the forces by which they are stretched, will be as the sides of the triangle BNC. For example, if BN = 4, NC = 3, BC = 5, the tensions of the cords AB, GB, BC, will be as 4, 3, and 5, respectively.

Corol. 3. The forces in the directions AB, GB may be reduced to a single force (*b*B) acting in a direction contrary to



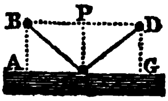
that of CB, and the body kept in equilibrio by two opposite and equal efforts. But if the body (B) be put in motion by three given forces PB, QB, RB of the same kind, acting in the same plane, then a single force equivalent to all three may be found thus: Complete the parallelogram QBPO, and the diagonal OB will represent a force equal to the two forces PB and QB; and if RS and OS are respectively parallel to BO and BR, the two forces OB and RB will be reduced to the diagonal SB or the single force SB, which therefore is the force equivalent to the three given forces; that is, the single force SB acting in the direction SB would have the same effect on the body B as the three given forces acting together in the directions PB, QB, and RB.

Hence it appears that any single force may be resolved into three or more forces acting in different directions.

SCHOLIUM. What is advanced in this last article will hold true in all kinds of forces whatever, whether of impulse or percussion, pushing, or drawing, or whether instantaneous, or continual, provided they are similar.

ON THE COLLISION OF BODIES.

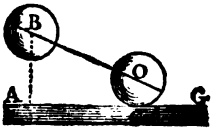
321. *If a perfectly elastic spherical body* B *impinge on an immovable plane* AG, *it will rebound or be reflected from the surface in an angle equal to the angle of incidence; that is, if* C *be the point of impact, the angle* DCG $=$ BCA.



Let BC denote the force of the body in that direction, which suppose to be resolved into two other forces BP and BA, the former parallel and the latter perpendicular to the plane AG; then if we conceive the body to be urged or carried along the diagonal BC by those forces or lines moving parallel to themselves, it must meet the plane in the point C with a force equal to PC; and since there is no resistance in the direction of the surface AG, the force (BA or PC) in that direction will not be retarded by the stroke, the body therefore, after impact, is urged by two forces respectively equal to the two former, namely, one in the direction of the surface CG as before, the other in that of CP, this latter is the re-acting or restoring force (*def.* 9) which, if the body be perfectly elastic, is equal and contrary to the compressing force PC (*ax.* 2 and 3); hence, by composition, CD the track of the body after impact, must be inclined to the reflecting surface in the same angle as before.

Corol. 1. The velocity with which the body quits the reflecting surface is equal to that at the time of impact, because the generating forces are equal and in similar directions.

Corol. 2. Since the times of compression and restitution are not instantaneous, the body is moved in the direction CG during those times by the force AC, and consequently the point of incidence and that of reflection cannot *accurately* be the same if the body is elastic.

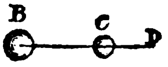


Corol. 3. When the surface AG is not smooth, the body will be reflected with a whirling motion: For let O be its center, and C the point of impact; then while the body is retarded in the direction AG by the friction at C, the force in the direction BO must produce a motion by which it endeavours to roll. This is confirmed by experience, for spherical bodies are seen to acquire a rotatory motion when reflected obliquely. This motion may affect the direction of the body when it quits the plane: And if the body is not perfectly elastic, the restoring force will be less than the compressing one: on these accounts, it is probable that the angles of incidence and reflection are always different.

Corol. 4. If the body be non-elastic, it will not acquire or generate a restoring force by impulse, (*def.* 9) it must therefore after the impact, be carried along the surface CG by the force acting in that direction.

Remark. If AG be a polished surface, and BC a ray of light proceeding from the lucid point B, the ray will be reflected in the direction CD, that is, the angles of incidence and reflection are equal in that case. This is a fundamental law of Optics, founded in nature according to some writers, because 'tis said *nature always acts by the most expeditious methods;* for the sum of the lines BC and DC are *less* than the sum of any other two lines that can be drawn from the points B and D to meet in the surface AG (*Theorem,* art. 272). Sir I. Newton however, has shewn that the reflection of light is not effected by its particles striking against bodies, but by some repelling power that extends beyond their surfaces. But if a particle of light moving along BC be struck in a direction (CP) perpendicular to the surface AG, either at C, or before it reaches that point, so that its velocity is not changed by the impulse, it will be reflected in an angle equal to that of incidence (*corol.* 1.)

322. *Suppose* B *and* C *are two equal non-elastic bodies, and let the body* B *strike the quiescent body* C *in the direction of their centers with a velocity of V feet per second, then after the impact they will proceed together as one body in the direction* CD *with a velocity equal to half V.*



For both bodies being non-elastic, they cannot generate any force that will cause them to recede from one another; and since a double quantity of matter is moved by the same force, (that of B at the impact) the velocity must be diminished in the same proportion (315), that is, the velocity is reciprocally as the augmented weight or mass;

$$B : V :: B + C \text{ (inversely) } : \frac{V \times B}{B + C}, \text{ or } \tfrac{1}{2} V \text{ when } C = B.$$

Corol. The momentum of both bodies moving together, will be the same as that of B before the stroke; for let B and C denote the weights or quantities of matter in the bodies B and C, whether equal or unequal: then $V \times B$ will represent the momentum of B, and $\frac{V \times B}{B+C} \times (B+C)$ or $V \times B$ that of both after the impact.

Suppose $B = 12$lb. $c = 4$lb. and $V = 20$ *feet*, then $\frac{V \times B}{B+C} = \frac{20 \times 12}{12+4}$ $= 15$ *feet* the velocity per second of both together after their congress.

323. *Let the body* B *moving with a velocity* $= V$ *overtake and strike the body* C *whose velocity in the same direction is* $= v$; *then if the bodies are non-elastic, they will proceed together as one mass with a velocity* $= \frac{V \times B + v \times C}{B+C}$.

For the force lost in B by the stroke is communicated to C, because action and re-action are equal; and therefore the force or momentum of both moving together is equal to the sum of the separate momenta, that is $V \times B + v \times C$; and this divided by the mass $B+C$ gives $\frac{V \times B + v \times C}{B+C}$ the velocity.

Let $B = 16$lb. $C = 4$lb. $V = 10$, and $v = 5$;

Then $\frac{V \times B + v \times C}{B+C} = \frac{10 \times 16 + 5 \times 4}{16+4} = 9$ *feet*, the velocity per second.

Corol. 1. If C be quiescent, $v = o$; and $\frac{V \times B}{B+C}$ is the velocity with which they proceed together after impact.

Corol. 2. If C moves in a contrary direction, or towards B, then $\frac{V \times B - v \times C}{B+C}$ will denote the velocity after the stroke. And when $V \times B - v \times C = o$, all motion is destroyed by the concourse. But if $V \times B - v \times C$ be negative, both bodies will move together towards B.

Suppose $B=4$, $C=16$, $V=5$, and $v=10$;

Then $\frac{V\times B-v\times C}{B+C}=\frac{5\times 4-10\times 16}{4+16}=-7$ the velocity, which therefore is towards B after impact.

Corol. 3. The velocity lost by B is $V-\frac{VB+vC}{B+C}=\frac{VC-vC}{B+C}$,

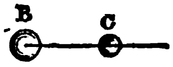
that is, $B+C : C :: V-v :$ velocity lost by B.

In like manner, $B+C : B :: V-v :$ velocity gained by C.

If the bodies move in contrary directions, that body must prevail whose momentum is greatest, but its velocity will be diminished, consequently $V-\frac{VB-vC}{B+C}$, or $\frac{VC+vC}{B+C}$ is the velocity lost,

That is, $B+C : C :: V+v :$ velosity lost by B, supposing it the most powerful of the two.

324. *If a non-elastic body* B *impinge directly on a fixed but perfectly elastic body* C *with a given velocity, it will rebound with the same velocity; and the whole force exerted by* C *against the striking body* B, *is double the force of impact when both bodies are non-elastic.*



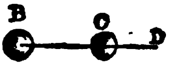
For if both were non-elastic, the motion or force of B would only be destroyed by the impact, or the bodies would adhere; but when C is perfectly elastic, it not only destroys all that motion or force but exerts another force equal and contrary to it in the action of recovering its figure before the stroke; consequently B will recede with its former velocity: and as the elastic body C first destroys and then restores the same force, its effect is double that of a non-elastic body. And if both bodies are perfectly elastic, the effect is the same, for the whole force of restitution must be equal to that of compression.

Corol. 1. If C be moveable, the velocity lost by B, and communicated to C by the stroke will be double what they would be were the bodies non-elastic; for the restoring force acts just as much in the direction of B's motion as against it, consequently while the bodies recede, B is retarded and C urged by additional forces equal to that of impact; that is, the velocities lost, or communicated by collision, are twice as great in elastic as in non-elastic bodies.

Corol. 2. Since the force lost in one body is gained by the other, if B and C are equal, and both perfectly elastic, C being moveable, the striking body B will rest after collision, and the other C move with a velocity equal to that of B before the impact.

Corol. 3. Hence it appears that the velocities are relatively the same before and after the impulse, that is, the bodies will be equally distant from one another at equal times before and after the impact.

325. *If the body* C *moving towards* D *with a celerity* $= v$ *be struck by the body* B *whose celerity in the same direction is* $= V$; *to find their velocities after the impulse, supposing both are perfectly elastic.*



It follows from *art.* 323 corol. 3, and *art.* 324 corol. 1, that the celerity lost by B after the impact, is $\frac{VC-vC}{B+C}\times 2$, and therefore $V-\frac{VC-vC}{B+C}\times 2$, or $\frac{V(B-C)+2vC}{B+C}$ is its velocity in the direction BD or DB, according as the expression is positive or negative.

And (by the same corollaries) $v+\frac{VB-vB}{B+C}\times 2$ or $\frac{v(C-B)+2VB}{B+C}$ is the velocity of C in the direction CD.

But if C be moving in a contrary direction, or towards B, then by making v negative, the same expressions become

$$\frac{V(B-C)-2vC}{B+C} \text{ the velocity of B ;}$$

$$\frac{v\,(B-C)+2VB}{B+C} \text{ the velocity of C,}$$

towards D when the expressions are positive, but in the opposite direction if they are negative.

Let B = 12*lb.* $V=5$ *feet* per second; C = 8*lb.* and $v=60$ *feet* per second; and suppose the bodies move in contrary directions,

$$\text{then } \frac{V(B-C)-2vC}{B+C}=\frac{5(12-8)-2\times 60\times 8}{12+8}=-47, \text{ velocity of B,}$$

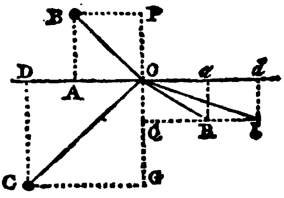
$$\frac{v\,(B-C)+2VB}{B+C}=\frac{60(12-8)+2\times 5\times 12}{12+8}=18, \text{ velocity of C;}$$

Therefore after collision, the bodies will move again in contrary directions, with velocities of 47, and 18 *feet* per second, respectively.

Corol. If C be at rest, then by making $v=0$, we shall get the velocities in that case.

The preceding method of investigation will answer when one of the bodies is non-elastic; or when both are imperfectly elastic, provided the forces of elasticity are known.

326. *If the non-elastic bodies* B *and* C *move in the same plane, and strike one another obliquely at the point* O *with given velocities; to determine their directions and velocities after collision.*



Let BO and CO be taken in the ratio of the respective velocities; and suppose D*d* is drawn to touch the bodies at their point of contact O. Complete the rectangles BAOP, CDOG; then the celerity BO is resolved into two others AO and PO, and the celerity CO into DO and GO (319, corol. 2): now as the efforts of the bodies against each other are made in the line joining their centers, those forces are not affected by the volocities (AO, DO) in the direction D*d*; consequently the velocities, in the line (PG) in which they act against one another, are denoted by PO and GO: and $\frac{B \times PO - C \times GO}{B + C}$ is the celerity with which they would proceed together after direct impact with the velocities PO and GO (323, corol. 1): If B be the most powerful, let OQ (in OG) be made $= \frac{B \times PO - C \times GO}{B + C}$, and take $Oa = OA$, and $Od = OD$, and complete the rectangles OQR*a*, OQI*d*; then the diagonals OR and OI will be the directions and velocities of B and C, respectively.

If the bodies are elastic, they will be reflected after impact; but the construction is no ways different: for having found the velocities in the line PG by art. 325, the result will point out whether they must be set off on the same, or on contrary sides of O.

N.B. The bodies are supposed to move along OR and OI after impact; strictly speaking however, their centers do not describe those diagonals, but lines parallel to them.

SCHOLIUM.

The preceding conclusions respecting the collision of bodies are confirmed by experiment, abstracting from the imperfection of materials; for it is probable there is no surface perfectly smooth, nor any hard bodies either perfectly elastic or non-elastic. Some experiments however, made with a view to ascertain the force of bodies in motion, seem to have misled several eminent mathematicians of the last century. Thus because it is found that a hard body impinging on soft and yielding substances of uniform consistence will penetrate to depths proportional to the squares of the velocities of impact, it has been infered that the momentum or force of bodies in motion, instead of being compounded of its velocity and mass (def. 4) is as the *square* of the velocity into the mass: this erroneous conclusion results from ascribing a whole effect to part of its cause: for the whole effect (or depth to which the body penetrates) is not produced by the motion or force of the body at the moment of impact, but by its successive efforts during the time of penetration, each effort being as the body drawn into the velocity with which it is moving. So bodies when projected vertically rise to heights proportional to the squares of the initial velocities (316, corol. 1) and during the time of ascent act against gravity, which, like the soft and yielding substances, is an uniformly retarding force; but to infer from this, that the force of the ascending body at any point of time is as the square of its velocity into the mass, would be contrary to theory and experiment.

Balls discharged from guns would penetrate wood, banks of earth, &c. to depths proportional to the squares of the velocities of impact, provided the resistances were uniform. And Mr. Robins found that musket bullets of equal size when shot against a block of elm with velocities of 1700, 730, and 400 *feet* per second, penetrated to the depths 5, $\frac{7}{8}$, and $\frac{1}{4}$ inches, respectively: these numbers are not exactly as the squares of the velocities; but "a greater coincidence cannot be expected when the unequal texture of the same piece of wood, and the change of the form of the bullet by the stroke are considered." (Gunnery, Chap. 2. Prop. 8). These experiments however, have been objected to as inconclusive *.

In estimating the force of a pile engine, the velocity of the weight or ram is easily determined: but if the pile be heavy, its momentum should be taken into consideration, because the ram and pile proceed as one body after the impact: and if the ground resist uniformly, the pile will sink to depths proportional to the squares of the velocities with which it begins to move..

Bodies impinging with equal momentums may have different effects. Thus a 48*lb.* shot with a velocity of 1000 *feet* per second, and a battering ram whose weight is 12000*lb.* moving with a velocity of 4 *feet* per second would have equal momentums, for $48 \times 1000 = 12000 \times 4$: but the former when discharged against a wall (for example) might pass through it without any other effect than that of driving out a few bricks or stones; whereas an impulse of the ram would probably cause a large breach: for that part of the wall upon which the ball impinges is separated and driven out before it can communicate much motion to the adjacent parts; but the shake is extended to a considerable distance by the slow movement of the battering ram, because the parts struck adhere together for a longer time.

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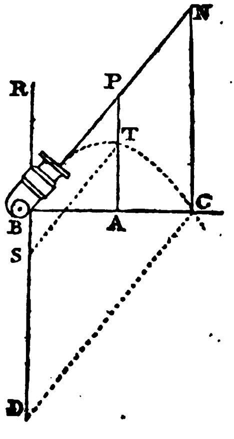
* Hutton's Math. and Philos. Dictionary, art. GUNNERY.

OF PROJECTILE MOTION.

327. *Let the equal lines* BD *and* CN *be perpendicular to the plane of the horizon represented by* BC; *and suppose a shell is discharged from the mortar* B *in the direction* BN *with a velocity that would carry it uniformly from* B *to* N *in the same time that a heavy body would descend by its gravity from* B *to* D; *then if the motion of the shell is not affected by the resistance of the air, it will describe the parabolic curve* BTC. (Art. 319, corol. 3).



BC is the horizontal range or amplitude.

CBN is the angle of elevation.

The velocity with which the shell quits the mortar, is the initial or projectile velocity.

And if the perpendicular BR be made equal to the height to which the body would ascend if projected vertically, it will represent the *impetus*.

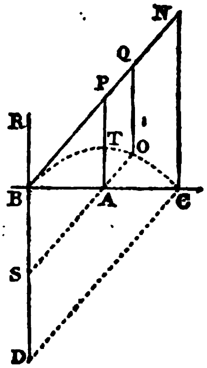
TA is the altitude of the projection, T being the highest point in the curve.

Thus suppose the angle of elevation CBN $= 45°$, the time of flight or that in which the shell describes the curve $= 12$ seconds $= t$, and $d = 16\frac{1}{12}$ *feet*; then $dt^2 = 2316$ *feet* $=$ BD or CN (317) $=$ BC the range in this case; and $dt^2\sqrt{2} =$ the tangent BN, which divided by t (the number of seconds) gives $dt\sqrt{2}$ *feet* the projectile velocity per second, hence (317) the vertical height to which it would ascend in t seconds is $\frac{d \times (dt\sqrt{2})^2}{4d^2}$ $= \frac{1}{2}dt^2$ the impetus BR; which therefore is $=$ half the range at an elevation of 45°.

Corol. 1. Let PA be perpendicular to the horizontal line BC, and TS parallel to PB; then the velocity of the projectile in the direction of gravitation at any point T, is to the projectile velocity in the direction BN, as 2BS or 2PT to PB. For BP and BS are described in the same time; but a body descending from rest through BS would acquire a velocity that would carry it uniformly through 2BS in the same time (317); and as the spaces described with uniform motions are as the velocities, therefore 2BS or 2PT is to BP, as the perpendicular velocity at T, to the projectile velocity in the direction BP.

Corol. 2. The horizontal celerity of the projectile is uniform: for the celerity along BN is uniform, and BA is directly as BP, by similar triangles. Hence also, because the velocity in the direction BC is constant, the celerity in the direction of the curve at any point (B) is as the secant of the angle of elevation; for BP is the secant to the radius BA. Therefore if T be the vertex of the parabola, the motion in direction of the curve will be slowest at that point; and the projectile will move with equal celerities at equal distances from it.

Corol. 3. Let SO, parallel to the tangent BN, bisect BD; then, as the velocity acquired in descending through BD is 2BD or twice the velocity of the projectile at B, therefore 2BS the velocity acquired at S, which is half that at D, will be equal to the projectile velocity; and by the first corol. as *velocity* in direction of gravity : *velocity* in direction BQ :: 2BS or 2QO : BQ; therefore 2BS or 2QO = BQ, because the velocities or two first terms of the proportion are equal. Whence (301) SO is the semi-parameter to the diameter BS: and when the elevation is 45°, A will be the focus of the parabola; and the height AT = $\frac{1}{4}$ of the range BC.



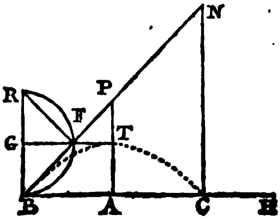
Corol. 4. Because, when S is the point where the celerity acquired by a body falling freely by the force of gravity from B, would be equal to the projectile celerity at B, the impetus BR is = BS, consequently $2BS = 2BR = 2QO = BQ = SO$, and $BQ^2 = 4BR^2 = 4QO^2 = BR \times 4QO$. But (299) PT, QO, NC, &c, are as BP^2, BQ^2, BN^2, &c. or $PT : QO :: BP^2 : BQ^2$,

$$\text{whence } BQ^2 = \frac{BP^2 \times QO}{PT} = BR \times 4QO, \text{ or } \frac{BP^2}{4PT} = BR,$$

that is, BR : BP :: BP : 4PT,
Also, BR : BQ :: BQ : 4QO,
BR : BN :: BN : 4NC, &c.

328. *Having the impetus, and elevation, to determine the random or horizontal range, and the greatest height to which the projectile will rise.*

If BH be the horizontal line, BR the impetus, and NBC the angle of elevation; then by the last of the preceding corollaries, we have to construct the right angled triangle BCN so, that BN is a mean proportional between BR and 4NC.



On BR describe a semicircle, and take $BN = 4BF$; let fall the perpendicular NC; and BC is the horizontal range.

Draw RF; then the triangles BCN, BFR being similar, we have

$$BR : BF \text{ (or } \tfrac{1}{4}BN) :: BN : NC,$$
$$\text{and } BR : \tfrac{1}{4}BN :: 4BN : 4NC,$$
$$\text{that is, } BR \times 4NC = BN^2,$$

Therefore BN is a mean proportional between BR and 4NC.

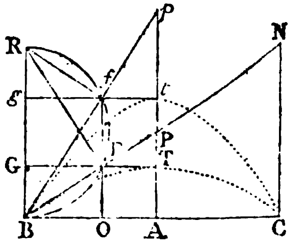
If BC be bisected by the perpendicular AP, and GFT drawn parallel to BC or perpendicular to BR, T will be the vertex of the parabola; and BG or AT its height above the horizon BC. For $PB = PN$, and $FP = FB$, and the triangles FBG, FPT being similar, BG is $= PT = AT$, therefore T is the vertex. (297, corol. 2)

And because $BN = 4BF$, the range $BC = 4GF$.

329. *Having the projectile velocity, and the distance* BC *of an object* C *on the horizontal plane; to find the angle of elevation of the mortar or cannon at* B, *so as to hit that object.*

If $v =$ the projectile velocity, and $d = 16\frac{1}{12}$ *feet*, then $\frac{d \times v^2}{(2d)^2}$ or $\frac{v^2}{4d}$ is the impetus BR. (327)

On BR describe a semicircle; take $BO = \frac{1}{4}BC$, and erect Of perpendicular to BC; then through F and f draw BN, Bp, and either of the angles CBN, CBp, is the elevation required.



For let the perpendicular Ap bisect BC, and draw gft parallel to BA; then it is proved that t is the vertex of the parabola BtC, in the same manner as T is found to be that of the parabola BTC.

Corol. 1. Hence there are two elevations which give the same range with the same velocity; one being as much above 45° as the other is below it.

Corol. 2. When BO or $\frac{1}{4}$ of the range BC, is $=$ the radius of the circle or $\frac{1}{2}$ the impetus BR, then O*f* will touch the circle, and the points F, *f*, coincide, in which case the elevation becomes 45°. The range therefore at 45° elevation is the greatest because its fourth BO will be a maximum.

330. Let s and c denote the *sine* and *cosine* of the angle of elevation.

r the horizontal range or amplitude BC.
h the greatest height AT or At.
m the impetus BR.
v the projectile velocity or the number of *feet* per second the projected body would describe with its first or greatest velocity.
t the time of flight.
$d = 16\frac{1}{12}$ *feet.*

Then from the similar triangles BFR, BCN, we have

rad. : BR :: *sin.* angle BRF : BF,

That is, $1 : m :: s : sm = \text{BF}$, and $4sm = \text{BN}$.

And in the triangle BCN

rad. : $4sm$ (BN) :: $c : 4csm = r = \text{BC}$ the range:

But $2cs$ is the *sine* of double the angle whose *sine* is s (246), therefore $4cs$ is twice the *sine* of double the elevation; consequently if $a =$ the *sine* of twice the elevation, $2am$ is the horizontal range, or $2am = r$. Hence the ranges with the same impetus, are as the sines of double the elevations: for let A denote the sine of twice any elevation, and R the corresponding range, then $2Am = R$, and $m = \frac{R}{2A}$; also $2am = r$, whence $m = \frac{r}{2a} = \frac{R}{2A}$, or $\frac{r}{a} = \frac{R}{A}$, that is $a : r :: A : R$.

331. If the elevation be the same, but the velocities different, the horizontal ranges are as the squares of the velocities. For let M be the impetus, R the corresponding horizontal range, and a the sine of double the angle of elevation, as above; then $2aM = R$, whence $a = \frac{R}{2M}$; also $2am = r$, and $a = \frac{r}{2m} = \frac{R}{2M}$, that is, $m : M :: r : R$; but if V be the velocity corresponding to the impetus M, then m being $= \frac{v^2}{4d}$, and $M = \frac{V^2}{4d}$ (327) we have $v^2 : V^2 :: r : R$.

332. If both elevations, and also the velocities, are different, the ranges are in the compound ratio of the squares of the velocities and the sines of double the angles of elevation. Thus, let A denote the sine of double any angle of elevation, M, V, and R the corresponding impetus, velocity, and range; then since $2AM = R$, and $2am = r$, we have $\frac{2AM}{2am} = \frac{R}{r}$, that is, $AM : am :: R : r$; but $M = \frac{V^2}{4d}$, and $m = \frac{v^2}{4d}$, whence by substitution $AV^2 : av^2 :: R : r$.

333. To determine the height, AT for example, we have $\mathrm{BF} = sm$ (330), whence

rad. $: sm :: s : s^2m = \mathrm{OF} = \mathrm{AT} = h$ the height; s being the sine of the elevation OBF to *rad.* 1. But if the time t be given, then $\frac{1}{2}dt^2 = m$, and the height $h = \frac{1}{2}dt^2s^2$.

334. From the preceding articles, we collect the following expressions, namely

$$m = \frac{v^2}{4d},\ r = 4csm = 2am,\ h = s^2m = \tfrac{1}{2}dt^2s^2;$$

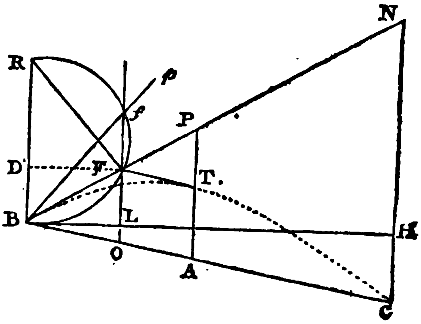
whence $v = \sqrt{4md}$, $m = \frac{r}{2a} = \frac{h}{s^2} = \frac{1}{2}dt^2$, $a = \frac{r}{2m}$,

$$s = \frac{r}{4cm} = \sqrt{\frac{h}{m}},\ t = \sqrt{\frac{2h}{ds^2}} = \sqrt{\frac{2m}{d}}.$$

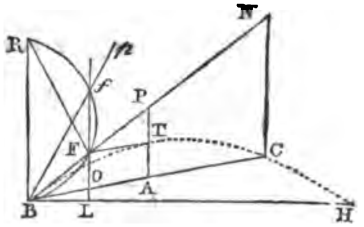
And by substitution, a variety of theorems may be found for the different cases on horizontal planes.

835. *Having the velocity, or impetus, and the angle of direction, to find the range on a plane inclined to the horizon.*

Let BH represent the horizontal line; BR (perpendicular to BH) the impetus; BC the oblique plane; and BN the direction of the projectile.



On the impetus BR describe the segment of a circle to contain an angle BFR equal to the supplement of RBC; take $BN = 4BF$, and draw NC perpendicular to BH; then BC is the range on the plane BC.



Join RF, and draw FO parallel to NC. Then since the angles BFR, BOF are equal, and the angle RBF equal to BFO, the angles FBO, FRB are therefore equal, and consequently the triangles BFR, BOF are similar: whence $BR : BF$ ($\frac{1}{4}BN$) $:: BN : NC$, and we have $BR \times 4NC = BN^2$ as in Art. 328; therefore BN being a tangent to the parabola at B, the curve will pass through the point C. And by sim. triang. $BC = 4BO$.

Corol. 1. If the impetus BR and range BC are given, the direction of the projectile is found thus: Let the circle be described as above; take $BO = \frac{1}{4}BC$, and draw OFf perpendicular to BH, then through F, f, draw BN, Bp, and either of those directions is that required, as in horizontal ranges.

Corol. 2. But if O*f* touch the circle, the points F, *f*, will coincide, and the direction bisects the angle RBC between the plane and impetus. And because in that case, BO is a maximum, therefore when the direction of the projectile is equally distant from the vertical BR and plane BC, the range BC will be the greatest possible, as in horizontal ranges.

336. Let S = the *sine* of FRB or FBO the angle of elevation above the plane.
C = the *sine* of BFR or BOF the *cosine* of the plane's inclination to the horizon.
c = the *sine* of RBF or BFO the *cosine* of the elevation above the horizon.
m = the impetus.
r = the range.
t = time of flight.
v = the velocity.
h = AT the greatest vertical height above the plane.

Then $C : \mathrm{BR} :: S : \frac{S \times \mathrm{BR}}{C} = \mathrm{BF}$,

$$C : \mathrm{BF} \left(\frac{S \times \mathrm{BR}}{C}\right) :: c : \frac{Sc \times \mathrm{BR}}{C^2} = \mathrm{BO}, \text{ and } \frac{4Sc \times \mathrm{BR}}{C^2}$$

the range BC, or $\frac{4Scm}{C^2} = r$.

337. Let AP, parallel to CN, bisect BC; then since TA is parallel to the axis of the parabola (which is perpendicular to BH), BC is a double ordinate to the diameter TA, therefore (298) AP is bisected in T, and FT (parallel to OA) a tangent to the curve at T, and consequently the triangles BFO, FPT are similar and equal;

hence, *sin.* BOF : $\frac{S \times \mathrm{BR}}{C}$ (BF) :: *sin.* FBO : FO;

That is $C : \frac{Sm}{C} :: S : \frac{S^2 m}{C^2} = \mathrm{FO} = \mathrm{AT} = h$ the greatest vertical height.

338. The time of describing the curve BTC is equal to the time that a body would be falling freely through NC or 4FO or $\frac{4S^2m}{C^2}$, (319 corol. 3); but if P be any space descended, then (317 schol.) $\sqrt{\frac{P}{16\frac{1}{12}}}$ is the time; therefore, putting $d = 16\frac{1}{12}$ *feet*, the time of flight will be $\sqrt{\frac{4S^2m}{dC^2}}$, or $\frac{2S}{C^2}\sqrt{\frac{m}{d}} = t$ seconds.

N. B. If S be taken for the *sine* of fBO the highest elevation, the computations refer to the upper parabolas; these however, are omitted in both figures.

339. The expressions $\frac{4Scm}{C^2} = r$, and $\frac{2S}{C}\sqrt{\frac{m}{d}} = t$

$$\text{give } m = \frac{C^2r}{4Sc} = \frac{dC^2t^2}{4S^2} = \text{(by art. 334) } \frac{v^2}{4d};$$

$$C = \sqrt{\frac{4Scm}{r}} = \frac{2S}{t}\sqrt{\frac{m}{d}};\ c = \frac{C^2r}{4Sm};$$

And by substitution we get the following theorems for the range, elevation, time, and velocity, on oblique planes:

$$r = \frac{Scv^2}{dC^2} = \frac{4Scm}{C^2} = \frac{dct^2}{S} = \frac{ct^2v^2}{4Sm}.$$

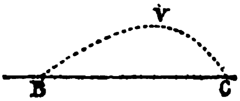
$$S = \frac{dtC}{v} = \frac{Ct}{2}\sqrt{\frac{d}{m}}.$$

$$t = \frac{2S}{C}\sqrt{\frac{m}{d}} = \frac{vS}{dC} = \frac{vrC}{4dcm} = \sqrt{\frac{Sr}{dc}}.$$

$$v = 2\sqrt{dm} = C\sqrt{\frac{dr}{Sc}} = \frac{dtC}{S} = \frac{4dtcm}{rC}.$$

340. The preceding deductions from the properties of the parabola however, are of little use in the practice of Artillery, on account of the very great resistance of the air, which in swift motions, is sometimes more than 20 times the weight of the projected body. And in consequence, the horizontal ranges are often less than $\frac{1}{5}$ of what they would be were the projections

made in vacuo. For example, if a cannon shot be discharged with an initial velocity of 1600 *feet* per second at 45° elevation, then $\frac{1600^2}{4 \times 16\frac{1}{12}} = 39793$ *feet* the impetus, and $2 \times 39793 = 79586$ *feet*, or upwards of 15 miles would be the horizontal range according to the parabolic theory; whereas in actual practice, it is found to be less than 3 miles. And the curve described is not at all similar to a parabola; its vertex or highest point, instead of being vertical to the middle of the range, is nearer the farther extremity, where the curve meets the horizon in a greater angle than that in which the body was projected. See the adjacent figure, where BC represents the horizontal line, and BVC the track described in the flight from B to C.



Many attempts have been made to investigate the nature of this curve, but from hypothetical data; and hence no theory has yet been found to agree with practice; but this will not be considered as extraordinary, since it is known by experiment that the same weight of shot, length of barrel, and quantity of powder frequently give different ranges.

Another great irregularity in the firing of shot is the deflection of the ball to the right or left of the mark. A deviation of this kind is likely to take place when there is considerable windage; for if the ball in its passage along the bore should touch one side, it will be reflected to the other, and again rebound to the opposite side, and so on, and thus acquire a kind of zig-zag motion: in which case the ball must quit the piece in a direction inclined to the axis of the bore. And the friction on that side of the mouth of the cannon touched by the shot when it quits it, will give the ball a whirling motion; the side of the ball therefore which moves foremost will be unequally resisted by the air in consequence of this rotatory motion; which is another cause of deflection to the right or left, except the axis of rotation be at right angles to the vertical plane in which the projection is made. Bullets discharged from a rifled barrel have the axis of rotation in the direction of the piece, and consequently that side of the bullet which moves foremost is equally resisted by the air in all its parts.

341. But the parabolic theory may sometimes be useful in slow motions if we employ data derived from good experiments, and proceed by comparison in circumstances not very dissimilar. Dr. Hutton found by experiments made at Woolwich that "shot which are of different weights and impelled by the firing of different quantities of powder acquire velocities which are directly as the square roots of the quantities of powder, and inversely as the square roots of the weights of the shot, nearly:"

That is, if p = the *lbs.* of powder,
w = the weight of the shot in *lbs.*
v = its initial velocity :

then if P, W, and V denote any other weight of powder, shot, and velocity, we have

$\frac{\sqrt{p}}{\sqrt{w}} : v :: \frac{\sqrt{P}}{\sqrt{W}} : V$, or $\frac{p}{w} : v^2 :: \frac{P}{W} : V^2$, which, when the balls are equal, or $W = w$, becomes $p : v^2 :: P : V^2$, that is, the squares of the velocities are as the quantities of powder, nearly. Which conclusion agrees with the experiments of Mr. Robins. Very small charges however, and such as exceed those that give the greatest velocity, are excepted.

342. Here follow some Examples in numbers.

1. If a ball of 1*lb.* acquire a velocity of 1600 *feet* per second when fired with 8 ounces of powder, what will be the velocity of a 13 inch shell weighing 196*lb.* when fired with 9*lb.* of powder?

Here $p = \frac{1}{2}$, $w = 1$, $v = 1600$, $P = 9$, $W = 196$;

and $\frac{p}{w} : v^2 :: \frac{P}{W} : \frac{wv^2P}{pW} = \frac{2304000}{98} = V^2$, and the square root $= 485$ *feet*; the *velocity* required.

2. If the horizontal range of a shell be a mile when discharged at 45° elevation, how far will it range when the elevation is 32°. 20′, the charge of powder being the same?

Here $a =$ *sin.* of twice 45°, $r = 1760$ yards, $A =$ *sin.* of twice 32°.20′; and $\frac{rA}{a} =$ the range, (art. 330);

$r = 1760$............. *log.* 3.2455
$A =$ *sin.* 64° 40′..... *log.* 9.9561
The *range*....... = 1591 yards *log.* 3.2016

Remark. Here it is supposed that the greatest range is at 45° elevation, but this will not be the case, except in very slow motions with great weight of shell or ball, for small shot discharged with considerable velocities are found to range the farthest when projected at about 30° elevation.

3. The horizontal range being 1760 yards at 45° elevation, then what must be the elevation with the same charge of powder to strike an object at the distance of 1591 yards?

In this example $r = 1760$, $R = 1591$, $a =$ *sine* of twice 45°, or the *sine* of 90°; and since $\frac{rA}{a} = R$ (in the preceding examp.) we have $A = \frac{aR}{r}$ the *sine* of double the required elevation:

$a =$ *sin.* 90°................. *log.* 10.0000
$R = 1591$..................... *log.* 3.2017
$r = 1760$........................ 6.7545 ar. comp.
sin. 60°.40′ or 115° 20′........ *log.* 9.9562

and the halves of 64° 40′ and 115° 20′ are 32° 20′ and 57° 40′ the required *elevations*. (329 corol. 1)

4. With what impetus, velocity, and charge of powder must a 13 inch shell be fired at an elevation of 32°.12′ to strike an object at the horizontal distance of 3250 feet?

If $r = 3250$, $a =$ *sin.* of twice 32° 12′, $m =$ the impetus, $v =$ the velocity, and $d = 16\frac{1}{12}$ *feet*:

Then (334) $m = \frac{r}{2a}$, and $v = \sqrt{4md}$.

$r = 3250$.................. *log.* 3.5119
$a =$ *sin.* 64° 24′......... *log.* 0.0449 ar. comp.
2............................ 9.6990 ar. comp.
Impetus 1802 *log.* 3.2558

4 ... . *log.* 0.6021
impetus .... *log.* 3.2558
$d = 16\frac{1}{12}$ .... *log.* 1.2064
2) 5.0643
Velocity 341 *log.* 2.5321

The charge is determined by comparing the velocity 341 with that of the shell when fired with a different quantity of powder: thus in examp. 1. the velocity with 9*lb.* is 485 feet,

Hence $485^2 : 341^2 :: 9lb. : 4.44lb.$ nearly, the *charge* required.

5. The horizontal range of a shell at 22° of elevation being 1400 yards, then how far will it range at an elevation of $29\frac{1}{2}°$ with the same charge of powder?

Here $a =$ *sin.* of twice 22°, $r = 1400$, and $A =$ *sine* of twice $29\frac{1}{2}°$; and $\frac{rA}{a} =$ the range, (art. 330);

$r = 1400$.................. *log.* 3.1461
$A =$ *sin.* 59° .... *log.* 9.9331
$a =$ *sin.* 44°.............. 0.1582 ar. comp.
Range = 1728 yards *log.* 3.2374

6. If the horizontal range of a shell be 1300 yards with 7*lb.* of powder, what charge will throw it 1000 yards, the elevation being 45° in both cases?

The squares of the velocities being nearly as the quantities of powder, we have (art. 331),

$1300 : 7lb. :: 1000 : 5\frac{5}{13}lb.$ the *answer.*

7. If the range of a shell when fired with 5*lb.* of powder at an elevation of 25° be 1600 yards on an horizontal plane, then how far will it range at an elevation of 50° when the charge is 4*lb.*

Since the velocities are nearly as the square roots of the quantities of powder, the squares of the velocities may be represented by 5 and 4:

Let $r = 1600$, $a =$ *sine* of twice 25°, $v^2 = 5$, $A =$ *sine* of twice 50°, $V^2 = 4$, and $R =$ the required range:

Then (332) $av^2 : AV^2 :: r : R$, or $\frac{rAV^2}{av^2} = R$.

$r = 1600$ .... *log.* 3.2041
$A = \sin.\ 100°$ .... *log.* 9.9933
$V^2 = 4$ .... *log.* 0.6021
$a = \sin.\ 50°$ .... 0.1157 ar. comp.
$v^2 = 5$ .... 9.3010 ar. comp.
Range = 1645 yards *log.* 3.2162

8. If the horizontal range of a shell at 34° elevation be 1100 yards, what is the time of flight?

Let m = the impetus, $a = \sin.\ 68°$ (twice the elev.), $r = 3300$ *feet* the range, $d = 16\frac{1}{12}$ *feet*, and t = the time of flight:

Then (334) $m = \frac{r}{2a}$, and $t = \sqrt{\frac{2m}{d}}$; and substituting $\frac{r}{2a}$ for m, we get $t = \sqrt{\frac{r}{ad}}$:

$r = 3300$ .... *log.* 3.5185
$a = \sin.\ 68°$ .... 0.0328 ar. comp.
$d = 16\frac{1}{12}$ .... 8.7936 ar. comp.
2) 2.3449
Time nearly = 14.9 seconds *log.* 1.1724

When the elevations is 45°, then $a = \sin.$ twice 45° = 1, and the expression $\sqrt{\frac{r}{ad}}$ becomes $\sqrt{\frac{r}{d}}$, and if d be taken = 16 (neglecting the fraction $\frac{1}{12}$) we shall have $\frac{1}{4}\sqrt{r}$ for the *time of flight*, nearly, in that case.

9. What will be the range of a shot on a plane which ascends 10° 20′, and on another which descends 10° 20′, the impetus being 2500 feet, and the elevation of the piece 34° above the horizon?

$$\left.\begin{array}{l} 34° - 10° 20' = 23° 40' \\ 34° + 10° 20' = 44° 20' \end{array}\right\} \text{elevations above the planes.}$$

If $S = \sin.\ 23° 40'$, $C = \cos.\ 10° 20'$, $c = \cos.\ 34°$, $m = 2500$, and r = the range:

Then (339) $\frac{4Scm}{C^2} = r$:

4	*log.*	0.6021	
S	*log.*	9.6036	
c	*log.*	9.9186	
m	*log.*	3.3979	
C^2		0.0142	ar. comp.
	log.	3.5364	.... 3439 feet, *range* on the *ascending* plane.

C	*log.*	9.9929
		2
C^2	*log.*	9.9858

And when $S = sin.\ 44°.\ 20'$, the same expression gives $r = 5987$ feet, the *range* on the *descending* plane.

10. What quantity of powder will throw a 13 inch shell 3439 feet on a plane which ascends 10° 20′, the mortar being elevated 34° above the horizon?

Here $r = 3439$, $d = 16\frac{1}{12}$; and $S = sin.\ 23°\ 40'$, $C = cos.\ 10°\ 20'$, $c = cos.\ 34°$ as in the last example:

And (339) $C\sqrt{\frac{dr}{Sc}} = v$ the velocity, or $\frac{drC^2}{Sc} = v^2$:

d	*log.*	1.2064	
r	*log.*	3.5364	
C^2	*log.*	9.9858	
S		0.3964	ar. comp.
c		0.0814	ar. comp.
2)		5.2064	
Velocity = 401 feet,	*log.*	2.6032	

Then, as in examp. 4.

$485^2 : 401^2 :: 9lb. : 6.15lb.$ nearly, the required charge.

11. In what time will a shell strike a plane which descends 7°, the impetus being 2000 feet, and the mortar elevated 45° above the horizon?

Let $S = sin.\ 45° + 7°$, $C = cos.\ 7°$, $m = 2000$, $d = 16\frac{1}{12}$ *feet:*

Then (339) $\frac{2S}{C}\sqrt{\frac{m}{d}} = t$ the time.

m	*log.*	3.3010	
d	*log.*	1.2064	
2)		2.0946	
		1.0473	
2	*log.*	0.3010	
$S = sin.\ 52°$	*log.*	9.8965	
C		0.0033	ar. comp.
Time = 17.7 seconds	*log.*	1.2481	

12. The time of flight of a shell was observed to be 14.4 seconds on a plane which ascends $8\frac{1}{2}°$; what was the elevation of the mortar, the impetus being 2304 feet?

Here $m = 2304$, $C = cos.\ 8° 30'$, $t = 14.4$, $d = 16\frac{1}{12}$:

And (339) $\frac{Ct}{2}\sqrt{\frac{d}{m}} = S$, or $\frac{C}{2}\sqrt{\frac{t^2d}{m}} = S$, the sine of the elevation above the plane.

t	*log.*	1.1584	
		2	
t^2	*log.*	2.3168	
d	*log.*	1.2064	
m		6.6375	ar. comp.
	2)	0.1607	
		0.0803	
C	*log.*	9.9952	
2		9.6990	ar. comp.
Elevation above plane 36° 30′	*log. sine*	9.7745	
8.30			
Elev. above horizon 45 0			

13. What must be the elevation of a mortar to throw a shell 6745 feet on a plane which descends 8° 15′, the impetus being 3000 feet?

Let $m = 3000$, $r = 6745$, $C = cos.\ 8° 15'$ or $sin.\ 81° 45'$, $T = tang.\ 8° 15'$.

Then $m : C :: \frac{1}{2}rC - mT$: *cosine* of an angle, half of which added to, and subtracted from half the supplement of 81° 45′, give two directions that will answer the question.

$\frac{1}{2}r = 3372\frac{1}{2}$	*log.*	3.5280		m	*log.*	3.4771
C	*log.*	9.9955		T	*log.*	9.1613
$\frac{1}{2}rC =$ 3337	*log.*	3.5235		$mT = 435$,	*log.*	2.6384
$mT =$ 435						
diff. 2902						

m		6.5229	ar. comp.
C	*log.*	9.9955	
2902	*log.*	3.4627	
cosine 16° 46′	*log.*	9.9811	

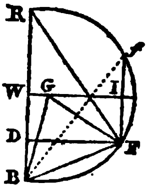
half 8 23

49 8 half the supplement of 81° 45′

sum 57 41 }
diff. 40 45 } the two required *elevations* above the plane, or 49° 26′, and 32° 30′ above the horizon.

For the construction of this case, see *art.* 335, corol. 1. But the proportion is investigated as follows:

Draw FD parallel to LB or perpendicular to BR (see the *fig.* art. 335); then since $BO = \frac{1}{4}BC = \frac{1}{4}r$, and $C =$ *sin.* angle BOL or BFR, therefore BL or $FD = \frac{1}{4}rC$. Hence in the triangle BFR we have given BR (the impetus), the opposite angle BFR, and the perpendicular FD, to determine the angles FRB, FBR.



Let the segment *Rf*FB contain the given angle BFR, and suppose G the center, and draw F*f* parallel to BR, and WI parallel to DF: then since the angle WBG is the difference between the angle BFR and a right one, it is equal to the inclination of the plane and horizon, and C is the *sine* of the angle BGW, and T the *tangent* of WBG; therefore $\frac{1}{2}m$ being $=$ WB, we have $\frac{1}{2}mT = GW$; but $DF = WI$, consequently $GI = \frac{1}{4}rC - \frac{1}{2}mT$; and by trigonometry,

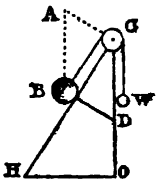
WB : *sin.* WGB :: BG : *radius*,
GI : *sin.* GFI :: GF : *radius*, but BG = GF, therefore
by equality WB : *sin.* WGB :: GI : *sin.* GFI,
that is $\frac{1}{2}m : C :: \frac{1}{4}rC - \frac{1}{2}mT$:
or $m : C :: \frac{1}{2}rC - mT$: *cosine* IGF, which angle being at the center, is $= f$BF the difference of the required angles FRB, FBR whose sum is the supplement of BFR.

When the plane is ascending, the center G will fall on the other side of BR, and GI the third term of the proportion will be $\frac{1}{4}rC + \frac{1}{2}mT$.

We have made use of logarithms because the computations are much shorter than by natural sines.

OF THE FORCE AND DESCENT OF BODIES ON INCLINED PLANES.—— MOTION OF PENDULUMS.

343. *Let* CO *be perpendicular to the horizon* HO, *and* CH *an oblique plane; and suppose the body* B *is sustained on the plane by the string* BC (*parallel to* CH) *fastened at* C; *then if* BD *be perpendicular to* CH, *the triangles* CBD, COH, *are similar*,



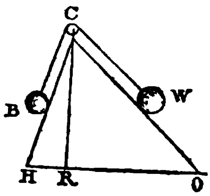
and the weight of the body B, *the tension of the string* CB *or the force of the body in the direction* CB, *the pressure against the plane* HC, *are respectively as* CD, CB, BD; *or* CH, CO, HO *the sides of the triangle* COH.

Let BA be parallel and equal to DC; then the force of gravity or the whole weight of the body acts in the direction AB or CD, the sustaining power in the direction HB or BC, and the opposing force of the plane in the direction DB; and since these three forces keep the body in equilibrio, they are as the sides of the triangle CBD (320) or COH.

Or because the sides of the triangle COH are as the sines of the opposite angles, the weight, the power in the direction BC, and the pressure on the plane, are as the radius, sine, and cosine of the plane's elevation above the horizon.

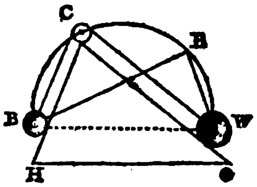
Suppose $HC = 5$, $CO = 4$, $HO = 3$, and the weight $B = 15$*lb.* then the sustaining force or power in the direction $BC = 12$*lb.* and the pressure against the plane $= 9$*lb.*

Corol. 1. Hence if the weight $B = 15$*lb.* be connected with another weight $W = 12$*lb.* by a flexible line BCW (considered as having no weight) that moves freely over a pin or pulley at C, the weight W acting in a perpendicular direction, will just prevent the other from descending along the plane, or the two weights will be in equilibrio.



Corol. 2. Hence also, if the two planes HC, OC are of an equal height above the horizon HO, and the weights B and W, connected by the line BCW moveable over the pulley C, are in the same proportion as the lengths HC and OC, the weights will mutually sustain each other on the planes.

Let HC $=5$, OC $=7$, the weight B $=15$*lb.* then $5 : 15 :: 7 : 21$*lb.* the weight W. And if the height CR $=4$, then HR $=3$, and OR $=\sqrt{33}$, and the pressure of B against the plane HC $=9$*lb.* and that of W against the plane OC $=\frac{21}{7}\sqrt{33}=17.23$*lb.* nearly.

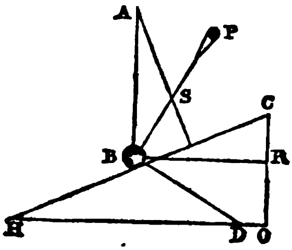


Corol. 3. If a circle be described through B, C, and W, the pulley, instead of being at C, may be at any point R in the arc BCW; and the weights B and W will remain in equilibrio when the connecting line is brought into the position BRW and its length $=$ BR + RW,

For the sustaining forces in the directions CB, CW being equal, and the angle CBR equal to CWR, it follows from the resolution of forces, that the weights are equally sustained in the directions BR and WR.

The angle CBR or CWR is called the angle of *traction.* And because BR is below the plane BC, the pressure of B against that plane is augmented, and that of W against CO diminished.

344. *Suppose the body or weight* B *is sustained on the plane* HC *by a string* BP *fastened at* P; *then if* BD *be perpendicular to* BP, *the weight* B, *sustaining force, and the pressure on the plane, will be respectively as* HD, BD, *and* HB.



Let BA be perpendicular to the horizon HO, and AS to the plane HC; then (320) the three forces acting on B, namely, that of gravity in the direction AB, the sustaining force in the direction BP, and the oppposing force of the plane in the direction SA, will be as the sides of the triangle BAS. But because AB and AS are perpendicular to HO and HC, and AB and BS to HO and BD, respectively, the angle BAS = BHD, and ABS = BDH, consequently the triangles HBD, ASB are similar, and the like sides proportional; that is, the forces are as the lines HD, DB, HB, drawn perpendicular to the directions of those forces.

Suppose HD = 5, HB = 4, BD = 2, and the weight B = 20*lb.*
Then 5 : 20 :: 4 : 16*lb.* the pressure on the plane,
5 : 20 :: 2 : 8*lb.* the sustaining force in the direction BP.

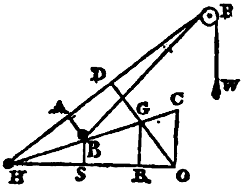
Corol. 1. The sustaining power (BP) is least when it is parallel to the plane BC; but greatest in the direction BA, for in that case it is equal to the weight B.

Corol. 2. But when the sustaining power (BR) acts parallel to the horizon, the weight of the body B, the sustaining power, and the pressure on the plane, are respectively as the base HO, perpendicular OC, and length of the plane HC.

For the sides of the triangle BCR or HCO are perpendicular to the directions of the three forces in equilibrio.

Let HO = 4, HC = 5, OC = 3, and the weight B = 20*lb.*
Then 4 : 20 :: 3 : 15*lb.* the sustaining power.
4 : 20 :: 5 : 25*lb.* the force against the plane.

345. *Let the body* B *upon an inclined plane* HC *be in equilibrio with another* W *hanging freely; then if they are put in motion, their perpendicular velocities will be reciprocally as their weights, or the weights multiplied by the respective velocities are equal.*



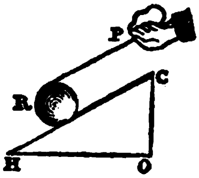
Let the weight B be made to descend a small distance on the plane from B to H. Draw OD, BA perpendicular to PH, and GR, BS perpendicular to the horizon HO; then BS will be the perpendicular descent of B; and because BH is supposed to be a very small space, AH may be considered as the difference of the lengths HP and BP (the two positions of the connecting line), which difference is the perpendicular ascent of the weight W. And since the triangles GRO, HBO are similar, and also the quadrilaterals HABS, HDGR, we have

GR : DH :: BS : AH
GR : DH :: GO : HO :: *weight* W : *weight* B (344)
whence by equality
BS : AH :: *weight* W : *weight* B, that is, the perpendicular velocities into the weights are equal.

Corol. Hence if two weights are in equilibrio on two inclined planes, their perpendicular velocities will be reciprocally as the masses when they are put in motion.

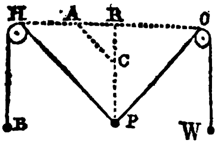
346. *If a cylinder* R *is sustained on the inclined plane* HC *by a power at* P *drawing one end of the rope* CRP *parallel to the plane, the other end being fixed at* C; *this power is to the weight of the cylinder, as half the height* CO, *is to the length of the plane* HC.

For (343) CH : CO :: *weight of cylinder* : *sustaining power in the direction of the plane*; but the sustaining power is equally divided between the parts of the rope RC and RP; therefore CH : $\frac{1}{2}$CO :: *weight of cylinder* : *sustaining power at* P.



SCHOLIUM. Hence it appears, that when the inclined plane is of sufficient length, a great weight may be raised to a given height by a small (comparative) force.

347. *Let* HO *be an horizontal line,* H *and* O *two pins or pullies,* B *and* W *two equal weights connected by a small flexible line* BHPOW *that moves freely over* H *and* O, *and* P *another weight at the middle of* HPO; *to find the length of the perpendicular* RP *when the three weights rest in equilibrio.*



Since the force of P sustains both the equal weights B and W, the force exerted by $\frac{1}{2}$P in the direction HP must be equal to the weight B; hence, by the resolution of forces, HP : RP :: force of $\frac{1}{2}$P in the direction HP : its force in the direction RP, therefore HP and PR have the same ratio as the weight B and $\frac{1}{2}$P; whence the following construction is obvious,

From any point C in the perpendicular RP draw CA so that $\frac{1}{2}$P : B :: CR : CA; make HP parallel to AC, and P is the place of the weight.

Since $AR = \sqrt{(AC^2 - RC^2)}$, we have, by sim. triangles,

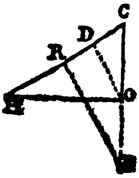
$$\sqrt{(AC^2 - RC^2)} : RC :: HR \left(= \tfrac{1}{2}HO\right) : \frac{RC \times HR}{\sqrt{(AC^2 - RC^2)}}$$

$= RP$, where AC and RC may be any quantities in the proportion of B and $\frac{1}{2}$P; therefore substituting B and $\frac{1}{2}$P for AC and RC, we get $\frac{P}{\sqrt{(4B^2 - P^2)}} \times \frac{1}{2}HO = RP$.

Suppose each of the equal weights $=$ 5*lb.* P $=$ 6*lb.* and HO $=$ 16 *feet*, then RP $=$ 6 *feet.*

Coral. If the weight P be equal to, or greater than both the other weights together, it will constantly descend, and consequently there can be no equilibrium.

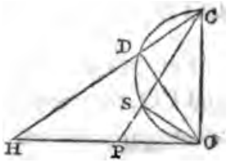
348. *If a body descend from rest along the inclined plane* CH, *the space it describes, is to the space it would describe in the same time when falling perpendicularly, as the height of the plane* OC, *to its length* CH.



The force with which the body endeavours to descend along the plane CH, is to the force by which it is urged in a perpendicular direction, as OC to CH (343): and since those forces are uniformly accelerating, the velocities acquired will be as the forces because the times are equal

(316), that is the velocities generated in the same time are as OC to CH; but (316 corol. 1) double the whole spaces described are as the velocities, and therefore the spaces are also as the velocities, or as OC to CH.

Corol. 1. Hence if CR = CO, and CB = CH, the triangles COH, CRB, are similar and equal, and therefore a body would descend from C to R in the same time that it would fall from C to B. Let OD be parallel to BR or perpendicular to HC, then by similar triangles, the times of descent through CD and CO are also equal.



Corol. 2. And if CP be another plane, and OS perpendicular to CP, then the time of descending from C to S is the same as that through the perpendicular CO; consequently the times along CD and CS are equal: but the *locus* of the right angles CDO, CSO, &c. is a semicircle described upon CO (Geom. 72); therefore the times of descent through all chords (CD, CS, DO, &c.) drawn from the extremities of the vertical diameter of a circle are equal, and equal to the time of the perpendicular descent through that diameter.

Corol. 3. The velocity acquired upon an inclined plane (CH) is to the velocity acquired in the same time by falling perpendicularly, as the height of the plane (CO) to its length (CH): Or as the sine of the plane's elevation to the radius.

349. *The time of descent along the plane* CH, *is to the time of falling through the perpendicular* CO, *as the length of the plane* CH, *to its height* CO. (see the last *fig.*)

Let T = the time of descent along CH, and t = the time in descending from C to D or from C to O. Then since the body is urged down the plane by an uniformly accelerating force, we have (316 corol. 1)

$$T^2 : t^2 :: CH : CD,$$

and $CH : CO :: CO : \frac{CO^2}{CH} = CD$, by sim. triangles;

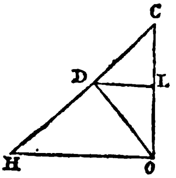
whence $T^2 : t^2 :: CH : \frac{CO^2}{CH}$,

or $T^2 : t^2 :: CH^2 : CO^2$,

That is $T : t :: CH : CO$.

Corol. Hence the times of descent along different planes of the same height, are as their lengths.

350. *A body acquires the same velocity in descending down an inclined plane* CH, *as by falling perpendicularly through* CO *the height of that plane.*



Let OD and DL be perpendicular to CH and CO, respectively. Then since CD is a mean proportional between CO and CL, it will be $CO : CL :: CD^2 : CL^2$ (Geom. 216) or $\sqrt{CO} : \sqrt{CL} :: CD : CL$.

Now CL being the height of the plane CD, we have (348, corol. 3)

$$\textit{veloc.}\text{ at } D : \textit{veloc.}\text{ at } O :: CL : CD,$$

$$\text{whence } \textit{veloc.}\text{ at } D = \frac{CL \times \textit{veloc.}\text{ at } O}{CD}.$$

Also because the velocities acquired at O and L are as the square roots of the heights CO and CL (316, corol. 1)

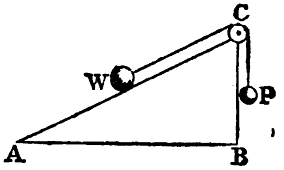
$$\textit{veloc.}\text{ at } O : \textit{veloc.}\text{ at } L :: \sqrt{CO} : \sqrt{CL} :: CD : CL,$$

therefore $\textit{veloc.}\text{ at } L = \frac{CL \times \textit{veloc.}\text{ at } O}{CD}$; that is, the velocity acquired at L in the perpendicular descent, is equal to the velocity acquired at D on the plane; and consequently the acquired velocities at H and O are also equal.

Corol. 1. Hence the velocities acquired by bodies descending from the same height, down any planes, to the same horizontal right line, are equal to one another.

Corol. 2. And the velocities acquired by descending down any planes, are as the square roots of the heights.

351. In Emerson's Mechanics, Prop. 31, we find the following Scholium, "If it be required to find the position of the plane AC, whose height BC is given; so that the given weight W may be raised through the length of the plane AC, in the least time possible, by any given power P, acting in direction WC. Make AC $= \frac{2W}{P} \times$ BC, and you have your desire." It may be worth while to show how this construction is derived.



Let the power P, denoted by a weight, descend perpendicularly and draw the weight W in the direction AC by means of a string passing over a pulley at C.

Put the height BC $= h$, $x =$ AC the required length, and $t =$ the time:

Then (343) AC : BC :: W : $\frac{BC \times W}{AC}$, or $\frac{hW}{x} =$ the power or relative weight which urges the weight W down the plane AC; and since P is the power or force in the direction CP, the difference $P - \frac{hW}{x}$, or $\frac{Px - hW}{x}$ will be the force which accelerates the bodies in motion. Now (317, corol.) the square of the time being as the space divided by the accelerating force, we have $t^2 \propto x \div \frac{Px - hW}{x}$, or $t^2 \propto \frac{x^2}{Px - hW}$; therefore the expression $\frac{x^2}{Px - hW}$ is to be the least possible or a *minimum*; consequently $\frac{x^2}{x - \frac{hW}{P}}$ must be the least possible, because in that case, any multiple or submultiple must also be the least possible.

Let $\frac{hW}{P} = g$; then $\frac{x^2}{x - g}$ must be a minimum.

Suppose $\frac{x^2}{x - g} = m$; then $x^2 = mx - mg$, and $x^2 - mx = -mg$, a quadratice quation, which completed is $x^2 - mx + \frac{1}{4}m^2 = \frac{1}{4}m^2 -$;

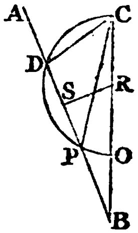
Whence $x = \frac{1}{2}m \pm \sqrt{(\frac{1}{4}m^2 - mg)}$, where if $\frac{1}{4}m^2$ be less than mg, the expression $\sqrt{(\frac{1}{4}m^2 - mg)}$ becomes impossible; therefore when the value of $\frac{1}{4}m^2$, as m is the least possible, $\frac{1}{4}m^2$ must be $= mg$, and $\sqrt{(\frac{1}{4}m^2 - mg)}$

$= o$, whence $\frac{1}{4}m = g$, and $\frac{1}{2}m = 2g$, and therefore $x = 2g$ (or $\frac{1}{2}m$) $\pm \surd(\frac{1}{4}m^2 - mg) = 2g \pm o$; that is $x = 2g = \frac{2hW}{P} = \frac{2W}{P} \times BC$.

Here the pulley is supposed to move without friction, and the string of no sensible weight.

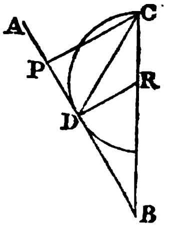
Corol. Hence it appears that the momentum of the power P, is double that of the weight W.

352. *If* C *be a point in the vertical line* CB; *to find the direction of the plane* CD *along which a body must descend from rest, to meet a plane or right line* AB *given in position, in a given time.*



Upon CO equal to the perpendicular distance which the body would descend in the given time, let a semicircle be described; draw CD and CP to the intersections D and P; and either of these directions is that required, as is manifest from art. 348 corol. 2.

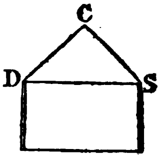
The distance CB and angle ABC are supposed to be given; hence if the perpendicular RS be let fall from the center R upon AB, the lines BP, BD, and angles PCB, DCB, are readily found by trigonometry.



353. If the circle should touch, instead of intersecting the line AB, then CD drawn to the point of contact D, will be the direction sought; and the time of descent from C to AB is the least possible, because the radius, or the diameter of the circle is the least possible. To construct this case: Upon AB let fall the perpendicular CP, bisect the angle PCB with CD, and draw DR parallel to PC; then R is the center, and RD the radius of the circle.

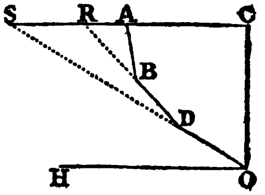
For the angle RDB is a right one, and therefore BD is a tangent to the circle; and because the angle RDC = PDC = DCR, the triangle DRC is isoceles, consequently RC = RD the radius.

Corol. 1. Hence when the time of descent is the least possible, the direction of the plane CD with the vertical CB, is equal to half the complement of the angle ABC which the plane or line AB makes with the vertical.



Corol. 2. And therefore if we would construct the roof of a house that the rain may run off in the least time, make the angle of the ridge DCS a right one, or each of the angles CDS, CSD equal to 45°.

354. *If a body descend from* A *to* O *down any number of contiguous planes* AB, BD, DO, *it will acquire the same velocity at* O *as a body falling perpendicularly the same height* CO, *provided the velocity is not altered by the different directions at* B, D, &c.



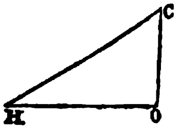
Draw CS parallel to the horizon HO, and produce the planes DB, OD, to meet CS. Then the velocities acquired in descending down the planes AB, RB, will be equal, because their heights are equal (350, corol. 1), and therefore the velocities acquired in the descent from R to D along the continued plane RD, is the same as that acquired in descending through both planes AB and BD; in like manner, the velocities generated in the descent along SO, and down the planes RD and DO, are also equal, and equal to that acquired in the perpendicular descent from C to O (350, corol. 1).

Corol. 1. If the lines AB, BD, DO, &c. are supposed to be diminished indefinitely in length, we may consider them as ultimately forming a curve line; and hence it is infered, that a body acquires the same velocity in descending along any curve, as in its descent through the same perpendicular height.

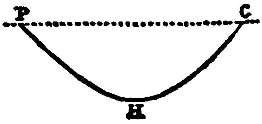
Corol. 2. Hence it also appears, that bodies acquire the same velocity in descending through any curves and planes of the same perpendicular height. And therefore when their velocities are equal at any particular height, they will be equal at all other equal heights.

Corol. 3. The velocities acquired by descending through any planes or curves, are as the square roots of the perpendicular heights (350, corol. 2).

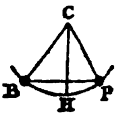
SCHOLIUM. In the preceding articles it is supposed that planes are perfectly smooth, or that bodies are not in the least retarded by friction in oblique descents. And since those motions are uniformly accelerated by the constant force of gravity, the descent on inclined planes is subjected to the laws already laid down for accelerated motion in a perpendicular direction (*art.* 316, 317). Thus, the velocities acquired in descending from rest along CH are as the times. And the spaces described as the squares of the times, or the squares of the acquired velocities. Also if the body were projected along the plane from H towards C with the velocity it acquired in the descent from C to H, the velocities and times of ascent would be equal to those of descent at equal distances from H.



Consequently, if a body descend freely along the curve CH, the velocity or force acquired at the lowest point H, would be sufficient to make it ascend up the curve HP, to the same perpendicular height, in the same time.

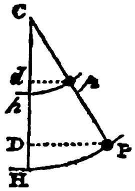


Hence if P be a body attached to one end of a small non-elastic string CP, the other end being fixed at C, and the body left to descend freely, it will describe the circular arc PHB; for the force acquired at the lowest point H would make it ascend to B in the horizontal line PB; and its motion being lost at that point, it



will descend back again, and rise to the same horizontal line BP in the same time: in this manner it would continue to oscillate or vibrate through the same arc BP by the force of gravity only, if all causes of resistance were removed. But experience proves that the air, and the friction, &c. at the center of motion or suspension C, act as retarding forces, by which means the vibrations are successively shortened till the body loses all motion at the lowest point H. A body moving in this manner is called a PENDULUM. And in theory, when all the matter of the pendulum or vibrating body is supposed to be in a single point (P), and the sustaining line CP without gravity, it is called a simple Pendulum. Pendulums that regulate the going of clocks continue their motion by the force of a weight, or spring, &c. called the maintaining power.

355. *If two pendulums* Cp, CP, *vibrate in similar arcs* ph, PH, *the times of vibration will be as the square roots of their lengths.*



Draw pd, PD, perpendicular to the vertical CH. Then (354, corol. 3) the velocities acquired at the lowest points h, H, in describing the arcs ph, PH, are as the square roots of the vertical heights hd, HD; but the sectors hCp, HCP, are similar, therefore the radii hC, HC, and also the arcs ph, PH, have respectively the same ratio as the homologous lines hd, HD; that is, the velocities acquired in descending from rest, are constantly as the square roots of ph, PH, the spaces described; and therefore the times of description are as $\sqrt{ph}$ and $\sqrt{PH}$, or as $\sqrt{Cp}$ and $\sqrt{CP}$.

Corol. Hence if T and t denote the respective times in which the pendulums CP, Cp, perform a vibration; then $\sqrt{CP} : \sqrt{Cp} :: T : t$, or $CP : Cp :: T^2 : t^2$, whence $Cp \times T^2 = CP \times t^2$, that is, the lengths are as the squares of the times of a vibration.

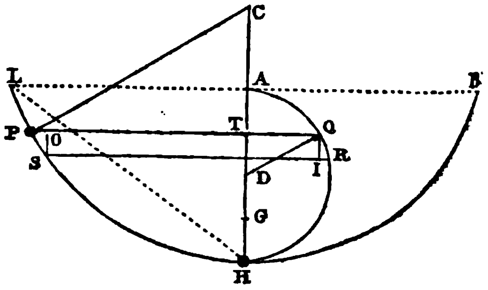
It is found by experiment that the length of a simple pendulum which vibrates once in a second of time in the latitude of London, is $39\frac{1}{8}$ *inches*, nearly; hence the length of a pendulum that vibrates once in $\frac{1}{2}$ a second is required?

The preceding expression gives $Cp = \frac{CP \times t^2}{T} = \frac{39\frac{1}{8} \times (\frac{1}{2})^2}{1^2} = 39\frac{1}{8} \times \frac{1}{4}$ $= 9.8$ *inches*, nearly.

If a musket ball be suspended by a very fine thread or wire, and the distance from the point of suspension to the center of the ball is 50 *inches*; how many times will it oscillate in a minute?

Here $t^2 = \frac{Cp \times T^2}{CP}$, and $t = \sqrt{\frac{Cp \times T^2}{CP}} = \sqrt{\frac{50}{39\frac{1}{8}}}$ seconds the time of one vibration, and $\sqrt{\frac{39\frac{1}{8} \times 60^2}{50}} = 53$ nearly, the number of vibrations or oscillations in 60 seconds.

356. But the vibrations in circular arcs of different lengths are not isochronal, that is, they are not performed in the same time. When however, the arcs are very small, the differences are found to be almost insensible. And in that case, *the time of one vibration, is to the time in which a body would descend perpendicularly through half the length of the pendulum, as the circumference of a circle, to its diameter.* (Emerson's Mechan.)



Let CH or CP be the pendulum vibrating in the arc LHB, C the point of suspension, and CH perpendicular to the horizon. On AH describe a a semicircle about the center D, and let PS be an indefinitely small part of the arc LPH; draw PQ, SR parallel to the horizontal line LB, and make SO, QI perpendicular to PQ, and join DQ.

Let t denote the time in which a body would descend perpendicularly through 2CH or along the chord LH. Then the velocities acquired in the descent through 2CH, and by the pendulum in falling through the arc LP, will be as the square roots of the vertical heights, or as $\sqrt{2CH}$ and $\sqrt{AT}$; and the space that would be described in the time t with the velocity acquired in falling through 2CH is 4CH, (316 corol. 1) which therefore may represent the velocity. And since the arc PS is indefinitely small, we may conceive it to be described with the uniform velocity denoted by $\sqrt{AT}$. But in uniform motions, the times of description are as the spaces divided by the velocities (314 corol. 1): hence $\frac{4CH}{\sqrt{2CH}} : t :: \frac{PS}{\sqrt{AT}} : \frac{t.PS}{2\sqrt{(2CH.AT)}}$, the time of describing PS.

If the arcs PS, QR, on account of their smallness, are considered as right lines, the triangles CPT, PSO, and DQT, QRT will be similar, respectively; whence CP : PT :: PS : $\frac{PT.PS}{CP}$, or $\frac{PT.PS}{CH} = OS = QI$; and DQ : TQ :: QR : $\frac{TQ.QR}{DQ}$, or $\frac{TQ.QR}{DH} = QI$, therefore $\frac{PT.TS}{CH} = \frac{TQ.QR}{DH}$; whence $PS = \frac{TQ.CH.QR}{PT.DH}$, which being substituted for PS in the expression $\frac{t.PS}{2\sqrt{(2CH.AT)}}$, and the time of describing PS is $= \frac{t.TQ.CH.QR}{2\sqrt{(2CH.AT)}\,(PT.DH)}$ or $\frac{t.TQ.CH.QR}{\sqrt{(2CH.AT)}\,(PT.2DH)}$.

But by prop. of the circle, $TQ = \sqrt{(AT.TH)}$. And $PT = \sqrt{[(CH+CT)TH]}$, these values of TQ, and PT being substituted, and we have the time of describing PS $= \frac{t\sqrt{(AT.TH)}.CH.QR}{\sqrt{(2CH.AT)}\sqrt{[(CH+CT)TH]}\,2DH}$, which reduced becomes $\frac{t\sqrt{2CH} \times QR}{\sqrt{(CH+CT)}\,4DH}$.

But $4DH = 2AH$. And $\sqrt{(CH + CT)} = \sqrt{(2CH - TH)}$; therefore by substitution, the time of describing PS $= \frac{t\sqrt{2CH}}{\sqrt{(2CH - TH)\,2AH}} \times QR$.

But if we suppose QR to be the arithmetical mean of all the QR's in the semicircle, the corresponding mean of the TH's will be DH or half AH, because the least TH is $= 0$, and the greatest $= AH$; hence the time of describing PS $= \frac{t\sqrt{2CH}}{\sqrt{(2CH - DH)\,2AH}} \times QR$. Now all the QR's is the semicircle ARH, and all the PS's the arc LPH; therefore the time of describing the arc $LPH = \frac{t\sqrt{2CH}}{\sqrt{(2CH - DH)\,2AH}} \times 2ARH$. And the time of one vibration along the arc $LHB = \frac{t\sqrt{2CH}}{\sqrt{(2CH - DH)\,2AH}} \times 2ARH$. But when the arc LHB is indefinitely small, DH may be taken $= 0$, and the expression becomes $\frac{t\sqrt{2CH}}{\sqrt{2CH \times 2AH}} \times 2ARH$, or $\frac{\frac{1}{2}t \times 2ARH}{AH}$. But when t is the time of perpendicular descent through 2CH, $\frac{1}{2}t$ is the time of descent through $\frac{1}{2}$ CH;

Therefore, as the diameter AH,

is to the circumference 2ARH,

so is $\frac{1}{2}t$ (the time of descent through half the length of the pendulum CH),

to $\frac{\frac{1}{2}t \times 2ARH}{AH}$ the time of one vibration in a very small arc.

Corol. 1. Hence the time of descent through a small arc (LPH), is to the time of descent along its chord (LH), as the diameter of a circle, to $\frac{1}{4}$ of its circumference.

For $\frac{\frac{1}{2}t \,.\, ARH}{AH}$ is the time of descent through the arc, and t the time along the chord,

And $AH : \frac{1}{2}ARH :: t : \frac{\frac{1}{2}t \,.\, ARH}{AH}$.

Corol. 2. If DH be bisected in G, and $T = \frac{t . \mathrm{ARH}}{\mathrm{AH}}$ the time of one vibration in a very small arc; then the time of vibration in any arc, will be $T + \frac{\mathrm{DG}}{\mathrm{CH}+\mathrm{CG}} \times T$, nearly.

For we found the time of vibration in LHB

$$= \frac{t . \sqrt{2\mathrm{CH} . 2\mathrm{ARH}}}{\sqrt{(2\mathrm{CH}-\mathrm{DH})2\mathrm{AH}}} = \frac{t . \mathrm{ARH}}{\mathrm{AH}} \sqrt{\frac{2\mathrm{CH}}{2\mathrm{CH}-\mathrm{DH}}} = T \sqrt{\frac{2\mathrm{CH}}{\mathrm{CH}+\mathrm{CD}}};$$

and the lines 2CH, CH + CG, and CH + CD, are in arithmetical progression, the common difference being GH; but if DH $= o$, they become equal, and therefore since DH is very small, they are nearly in geometrical progression; hence

$$\sqrt{\frac{2\mathrm{CH}}{\mathrm{CH}+\mathrm{CG}}} = \frac{\mathrm{CH}+\mathrm{CG}}{\mathrm{CH}+\mathrm{CD}},$$

therefore the time of vibration

$$= T \times \frac{\mathrm{CH}+\mathrm{CG}}{\mathrm{CH}+\mathrm{CD}} = T \times \frac{\mathrm{CH}+\mathrm{CD}+\mathrm{DG}}{\mathrm{CH}+\mathrm{CD}} = T \times \frac{\mathrm{CH}+\mathrm{CD}}{\mathrm{CH}+\mathrm{CD}} + \frac{\mathrm{DG}}{\mathrm{CH}+\mathrm{CD}} \times T$$
$$= T + \frac{\mathrm{DG}}{\mathrm{CH}+\mathrm{CD}} \times T.$$

Corol. 3. Hence also, we can determine the perpendicular descent of a body near the earth's surface in a second of time.

For it has been found by experiment that a simple pendulum $39\frac{1}{8}$ or rather 39.13 inches long, vibrates once in a second of time in the latitude of London. Therefore, putting $c = 3.1416$ the circumference of a circle whose diameter is 1, $p = 39.13$, $t =$ the time of perpendicular descent through $\frac{1}{2}p$, and $d =$ the distance required:

Then, 1 *sec.* : t :: c : 1, whence $t = \frac{1}{c}$ the time of descent through $\frac{1}{2}p$.

And the spaces described being as the squares of the times of description (316 corol. 1), we have $\frac{1}{c^2}$: $\frac{1}{2}p$ :: (1 *sec.*)^2 : d, or $\frac{1}{2}pc^2 = d = \frac{39.13}{2} \times 3.1416^2 = 193.1$ *inches*, nearly, $= 16\frac{1}{12}$ *feet*; for the descent of gravity in 1 second of time. Which agrees with experiment.

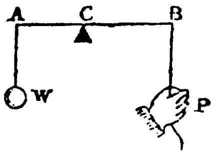
Remark. Pendulums of the same length vibrate quicker in remote latitudes than near the equator. For, by reason of the spheroidal figure of the earth, the distance from its center increases and therefore gravity decreases, as we approach the equator. But the greatest diminution arises from the rotation of the earth on its axis: for the parts near the equator move quicker, and thereby have a greater tendency to recede from the axis of motion, than bodies at the surface in other latitudes. Gravity, therefore, is greatest at the poles, and least at the equator.

Of the MECHANICAL POWERS.

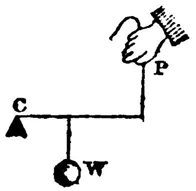
357. These are certain machines or instruments contrived for moving great weights with small force. They are commonly reckoned six in number; namely, the Lever, the Wheel and Axle, the Pulley, the Wedge, the Screw, and the Inclined Plane.

The Lever.

358. A lever is a rod or bar of wood or metal, as a hand-spike, a crow-bar, the beam of a pair of scales, &c. It is usually represented by an inflexible line without weight to render the demonstrations more simple. There are four kinds of levers.

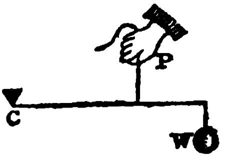


A lever of the first kind has the fulcrum or prop between the weight and the power. Thus if AB be a rod or bar, W a weight attached to the end A; P a power acting at the other end B, and C the fulcrum or prop that supports AB. Then C is the center of motion, and AC, and BC are the arms of the lever AB. Of this kind are balances, scales, scissars, pincers, &c.



A lever of the second kind has the weight W between the power P and the fulcrum C. As oars and rudders, bellows, cutting knives fixed at one end, &c.

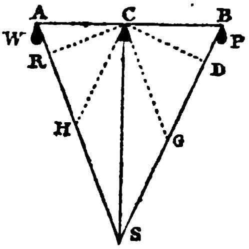
A lever of the third kind has the power P between the weight W and the fulcrum C. Such as tongs, sheep-shears, a man raising a ladder, the bones and muscles of animals, (Borelli de Motu Animalium).



A bended lever is the fourth kind. As a claw-hammer drawing a nail. This however, is only a species of the first kind, because the fulcrum is between the power and the body to be moved.

359. *Let the weights* W *and* P *be attached to the ends of the inflexible line or lever* AB, *and suppose* C *is the fulcrum or prop supporting the weights; then if they are in equilibrio, the distances* AC *and* CB *will be reciprocally as the weights; that is,* AC : CB :: P : W, *or* AC × W = CB × P.

Let S be the center of the earth. Then because the weights W and P gravitate towards the center S, the three forces in equilibrio act in the directions AS, BS, and SC. Draw CR, CD perpendicular to AS, BS, respectively, and CH, CG parallel to BS, AS; and the three forces will be as the sides of the triangle SCG, or CHS (320);

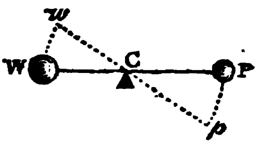


hence CH : HS (or CG) :: P : W.

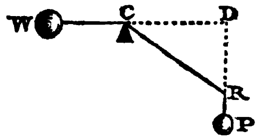
But in the parallelogram CHSG, the opposite angles at H and G are equal, consequently the angles CHR, CGD are also equal, and therefore the triangles CHR, CGD are similar,

whence CH : CG :: CR : CD.
But CH : CG :: P : W,
therefore by equality, CR : CD :: P : W;
or CA : CB :: P : W,

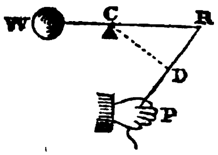
For CA, and CB, and the respective perpendiculars CR, and CD, are not sensibly different either in length or position.



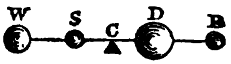
Corol. 1. Hence if the weights W, P, or the weight W and power P in equilibrio, move on the fulcrum or center of motion C, the arcs or spaces W*w*, P*p*, described in the same time, will be as the radii CW, CP, therefore the weight $W \times Ww$ $=$ power (or weight) $P \times Pp$; and since the velocities will be as the arcs W*w*, P*p*, the momenta of W and P are equal.



Corol. 2. And if WCR be a bended lever, and the power P act perpendicular to WCD, the weight W and power P will be in equilibrio when WC, DC, the perpendicular distances from the center of motion C, are reciprocally as the weight and power: that is, WC : DC :: P : W.



Corol. 3. Therefore when the power P acts obliquely against the end of the lever WR, the weight W and power P are reciprocally as WC and the perpendicular CD, the two distances of the directions of the forces from the center of motion C; that is WC : DC :: P : W. Hence, if WCD be a bended lever, and the weight W, and the power P, act perpendicularly to the arms CW, CD, then $WC \times W = CD \times P$, as in the straight lever.



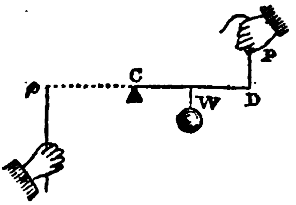
Corol. 4. When several weights W, S, D, B, acting on a straight lever WB, are in equilibrio, the sum of the products of the weights multiplied by their respective distances from the support C on one side, will be equal to the sum of the products on the other:

that is, $SC \times S + WC \times W = DC \times D + BC \times B$.

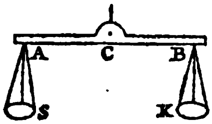
For the effort of each weight to turn the lever, is as the weight multiplied by its distance from the fulcrum C, and therefore the sum of the efforts on one side must be equal to those on the other, in the case of an equilibrium.

Corol. 5. Hence the place of the fulcrum is readily determined when the length of the lever WP, and the weights W, P, are given (see *fig.* to corol. 1). For $W : P :: CP : CW$; and by composition, $W + P : WP$ (the length) $:: P : CW$; that is, the length must be divided into two parts having the proportion of the weights.

360. A lever of the second, or third kind, may be reduced to the first, thus,



Conceive the lever Cp to be equal to CD, then it is manifest, that if the power P were removed to p, but acted in a contrary direction, the equilibrium would still remain, and we should have $pC \times p = CW \times W$, that is, $DC \times P = CW \times W$. Hence, in a lever of either kind, if the weight, and the power, are multiplied by their respective distances from the fulcrum, the products will be equal when there is an equilibrium.

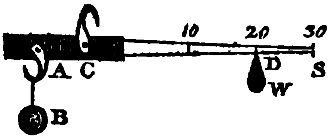


Scholium. The beam of a pair of Scales is a lever of the first kind. Its arms CA, CB ought to be exactly of the same length; for should there be any difference, equal weights when placed in the scales S and K, will not rest in equilibrio. The obvious method of trial however, is to weigh any body, very accurately, in one scale, then if the weight and body change places, and either end preponderates, the scales are imperfect or false.

But when we know what the body weighs in each scale, its true weight may be found thus——Let W and w denote the weights of the body b in the scales S and K respectively,

$$\text{then } CA \times b = CB \times W,$$
$$\text{and } CB \times b = CA \times w,$$

whence $b^2 = Ww$, or $b = \sqrt{Ww}$; that is, its true weight is a geometrical mean between the least and greatest weights found by the false scales.

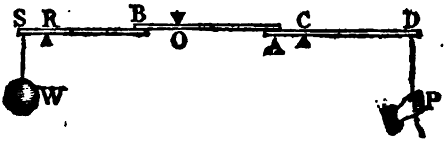


361. The steelyard or Roman balance is also a lever of the first kind. But the arms or *brachia* CA, CS are very unequal in length. The weight or counterpoise W is moveable backwards and forwards on the arm CS, which is divided and numbered. The distances of the divisions (which are notches in the beam) from the fulcrum C, are determined by repeated trials. Thus for example, suppose the weights B and W are in equilibrio, then if the weight B is 20*lb.* a notch is made in the beam at D, and that division is numbered 20. And in like manner, by suspending different weights at A, the other divisions are found.

If the arm CS be three quarters of a yard long, and CA one inch, then (neglecting the weight of the beam) a weight (W) of 2*lb.* at S, would weigh the body B of 54*lb.* For $27 \times 2 = 54 \times 1$.

362. *Let the compound lever* SD *be composed of three levers of the first kind,* DA, AB, BS, *acting upon one another*; *the fulcrums being at* C, O, R;

Then $P : W :: CA.OB.RS : CD.OA.RB$, *when the power P and weight W are in equilibrio.*



For $CA : CD :: P : \frac{CD.P}{CA}$ the force at A;

$$OB : OA :: \frac{CD.P}{CA} : \frac{CD.OA.P}{CA.OB} \text{ the force at B;}$$

$$RS : RB :: \frac{CD.OA.P}{CA.OB} : \frac{CD.OA.RB.P}{CA.OB.RS} = W, \text{ the}$$

the force at S;

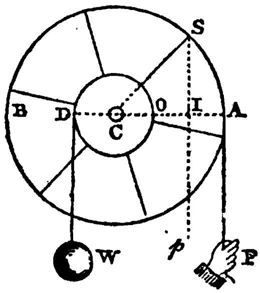
Therefore $CD.OA.RB.P = CA.OB.RS.W.$

And a similar conclusion is derived in the other kinds of levers, by making use of the respective distances from the props or fulcrums.

The heavy lever will be considered when we treat of the Center of Gravity.

WHEEL AND AXLE.

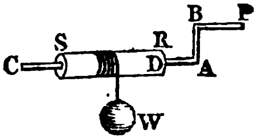
363. THIS instrument is a wheel AB fixed on a roller or axle OD, the axle being supported at its extremities so as to turn round freely with the wheel. It may be considered as a perpetual lever of the first kind. For when the weight W attached to a rope DW that goes round the axle, and the power P applied at the circumference of the wheel, are in equilibrio, then, as AC the radius of the wheel : CD the radius of the axle :: W : P, or CD×W=CA×P, as in the lever.



This will be obvious by considering the radii AC, CD as forming one line or lever AD, and C the fulcrum or center of motion upon which it turns.

Corol. 1. While the weight W, is drawn up by the power P, their velocities will be as the radii of the axle, and wheel, respectively.

Corol. 2. When the direction (S*p*) of the power applied to the wheel, is not perpendicular to its diameter, the radii SC, CD form a bended lever SCD; and if CI be drawn at right angles to S*p*, we shall have CI : CD :: W : power at *p*.



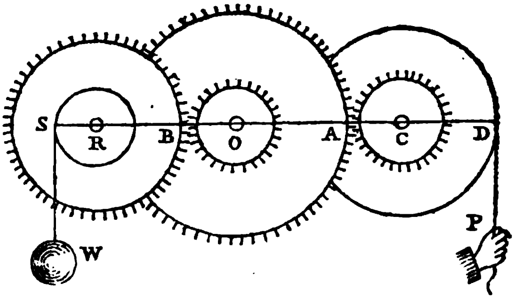
Corol. 3. If a roller or cylinder SD is turned on the axis CA by means of the handle ABP, and the power at P acting perpendicularly to AB, and the weight W, are in equilibrio; then P : W :: DR the radius of the roller : AB the length of the handle. For when the roller is turned round, the point B describes a circle whose radius is AB.

SCHOLIUM. But the weight and thickness of the rope to which the body (W) is appended, ought to be taken into the account. Thus, every fold of rope on the axle or roller, may be said to increase its diameter; the radius therefore is always the distance from the axis of the roller to the middle of the outside rope or fold; for which reason it will sometimes be found necessary to increase the force. The weight however, will evidently diminish as the rope (which makes part of the weight) shortens, or is wound on the axle.

To the wheel and axle may be refered several kinds of machines or instruments, as the crane, windlass, capstan, gimblet, auger, &c.

364. *In a combination of wheels with teeth, if the power* P *be to the weight* W, *as the product of the radii of all the axles or pinions, to the product of the radii of all the wheels; the power and weight will be in equilibrio.*

That is, P : W :: CA.OB.RS : CD.OA.RB.



For the radii CD, OA, RB, of the wheels, with the radii CA, OB, RS of the pinions (or smaller wheels on the axes of the larger) act as three levers DA, AB, BS on the centers or fulcra C, O, R; therefore the three together may be considered as a compound lever SD (art. 362).

Corol. 1. And when the wheels are in motion, the velocity of the power P : velocity of the weight W :: CD.OA.RB : CA.OB.RS. Or the number of teeth in the wheels and pinions may be substituted for the respective radii.

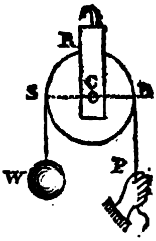
Corol. 2. Hence also, as the number of revolutions of the first wheel, is to the number of revolutions of the last wheel in the same time, so is the product of the number of teeth in the pinions, to the product of the number of teeth in the wheels. For as often as the number of teeth in any pinion, is contained in the number of teeth of the wheel that drives it, so many revolutions will the pinion make, for one revolution of the wheel.

Suppose the radii of the pinions are each $= 4$, CD $= 25$, OA $= 32$, RB $= 25$, and the power P $= 10$*lb.*

Then $4 \times 4 \times 4 : 25 \times 32 \times 25 :: 10 : 3125$*lb.* the weight W when the weight and power are in equilibrio. But were the wheels in motion, the velocities of the power and weight would be as 3125 to 10.

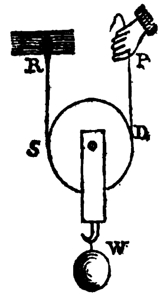
By the addition of another wheel and pinion, it is manifest a much greater weight might be raised by the same power, but the velocity of the weight would be proportionably diminished : Hence the maxim in Mechanics, *what is gained by power is lost in time.*

THE PULLEY.



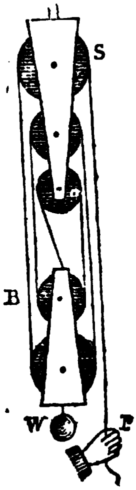
365. A PULLEY is a small wheel SD of metal or wood, moveable round an axis C fixed in a block; a groove is cut round the edge of the wheel to receive the rope WSRDP.

If the pulley is fixed, and the weight W *and power* P *are in equilibrio, the power and weight will be equal.* For SCD is a lever of the first kind, where the arms CS, CD are equal.



When the pulley, and weight W, are supported by the rope RSDP, the power at P will be but half the weight in case of an equilibrium. For the rope being fastened at R, the two parts RS, PD will be equally stretched, and consequently each will bear half the weight.

366. *In a combination of pullies drawn by one rope going over all the pullies, if the power* P, *is to the weight* W, *as* 1, *is to the number of parts of the rope proceeding from the moveable or lower block and pullies* (B); *then the power and weight will be in equilibrio.*



For the lower or moveable block (B) and the weight W together with the power P, are supported by 6 ropes, or rather 6 parts of the same rope, all equally stretched; but the part SP sustains the power P, and therefore each of the other parts is stretched by $\frac{1}{5}$ of the weight, consequently (in the annexed figure) 5 : 1 :: W : P.

Corol. Hence if the weight W be raised by the power P, the latter will descend 5 feet (for example) while the former is raised 1 foot. For each of the parts of the rope proceeding from the lower or moveable block to the upper or fixed one, is shortened by 1 foot, and consequently the weight will be raised that distance.

There are various other combinations of blocks with pullies, but the equilibrium is determined in a similar manner; namely, by comparing the spaces described in the same time by the power, and the weight or body moved.

But pullies, and other machines seldom work without considerable friction; and ropes are never perfectly flexible; therefore we must not expect that computations will always agree with experiment.

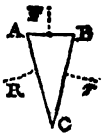
OF THE WEDGE.

367. THE form of this instrument is sufficiently known (art. 280, vol. 1). It is commonly made of wood or iron; and used in splitting blocks of stone, wood, &c. and sometimes in lifting or raising very heavy bodies. The force is communicated to the wedge by the blow of a sledge-hammer or heavy mallet.

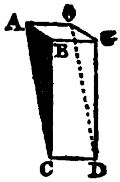
When the force F *acting perpendicularly to the back of the wedge* AB, *is in equilibrio with the resistances* R, *r*, *which act perpendicular to the sides* AC, BC,

Then AB : F :: AC + BC : R + *r*.

For the three forces in equilibrio will be as the sides of the triangle ABC which are perpendicular to the directions of those forces, (320):



That is AC : BC :: R : *r*;
or AC + BC : BC :: R + *r* : *r*.
Also AB : F :: BC : *r*;
And by equality AB : F :: AC + BC : R + *r*.



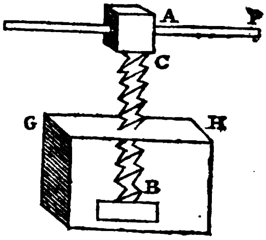
368. If the wedge be rectangular, or the triangular sides ABC, OGD perpendicular to the back or end ABGO, those sides are parallel to the direction of the force (F) acting against the end AG; and therefore any resistances against the planes ABC, OGD would have no effect in producing an equilibrium except what arose from friction. The friction however, on the quadrangular sides only is so very great that it retains the wedge in its situation after the force (F) is removed, and hence we must conclude that it is, at least, equal to the force which drives the wedge.

But we cannot compare percussive force with weight or pressure. A common iron wedge is sometimes wholly forced into a block of tough wood without splitting it, by a few blows with a heavy mallet; now it is impossible to discover by calculation, and it would be extremely difficult to determine from experiment, the enormous weight or pressure necessary for producing a like effect.

The wedge is a very simple mechanic power; and to this may be reduced most edge tools, and those that have a point; as the axe, chissel, spade, &c. and nails, bodkins, needles, &c.

Of the SCREW.

369. The screw is a cylinder CB round which is cut a spiral groove; the part that rises above the groove also forms a spiral, and is called the thread or threads of the screw; these make the same, or equal angles with the length BC.



When the screw is made use of as a press, &c. CB is called the male screw; the female, in which the other turns, is concave or hollow, and fixed in a block or frame GH.

If the screw be turned by a power P acting at the end of a lever AP; then (abstracting from friction), *As the distance between any two contiguous threads measured in the direction of the length* CB, *is to the circumference of the circle which the power* P *describes, so is the power* P, *to the force at* B, *when there is an equilibrium.*

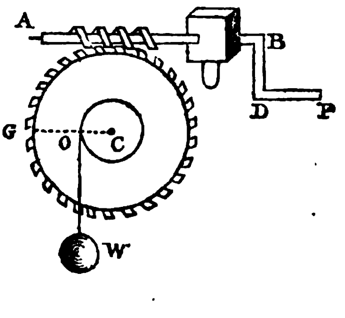
For the screw moves the distance between two threads in the direction CB at every revolution; and in case of an equilibrium, the forces (at P and B) are reciprocally as the spaces described in the same time: Which is a general property in all mechanical powers.

Suppose the force at $P = 50$*lb.* the distance of the threads $= \frac{1}{3}$ of an inch, and $AP = 9$ *feet.* Then $18 \times 12 \times 3.1416 = 678.58$ *inches* the circumference described by P;

And $\frac{1}{3} : 678.58 :: 50 : 101787$*lb.* the force exerted by the lower end of the screw.

But the friction is generally so great that it prevents the screw from receding when the power (P) ceases to act; and therefore it would probably require a force $= 2P$ on the lever to produce the computed effect.

370. The endless or perpetual screw AB has square threads adapted to the spaces between the teeth of a wheel, which are cut oblique to fit the spiral groove whose sides are perpendicular to the axis of the screw. The screw is turned by means of a handle BDP.



Suppose the weight W to be supported on the pinion or roller whose radius is CO; and let n denote the number of teeth in the wheel, $C =$ its circumference, and $c =$ the circumference of the circle described by the handle or power P.

Then since the wheel is moved forward by one tooth of the wheel at every revolution of the screw or handle, P will make n revolutions while the wheel makes 1, or the power P moves through the spaces nc while the teeth describes the circumference C:

Hence, *velocity* of P : *vel.* of teeth :: $nc : C :: n \times BD : CG$, (because the circumferences c, C, are as the radii BD, CG).

But $CO : CG ::$ *vel.* of W : $\frac{CG \times vel.\ of\ W}{CO}$ the velocity of the point G or of the teeth;

hence by equality, *vel.* of P : $\frac{CG \times vel.\ of\ W}{CO} :: n \times BD : CG$,

$$\text{or } vel.\ \text{of P} : \frac{vel.\ \text{of W}}{CO} :: n \times BD : 1;$$

$$\text{That is, } vel.\ \text{of P} : vel.\ \text{of W} :: n \times BD : CO:$$

But in the case of an equilibrium, the weight and power are reciprocally as their velocities,

$$\text{Therefore W} : P :: n \times BD : CO, \text{ or } W = \frac{P \times n \times BD}{CO}.$$

Let $BD = 12$ *inches*, $CO = 3$ *inches*, the number of teeth in the wheel $= 100$, and $P = 40lb.$

Then $\frac{40 \times 100 \times 12}{3} = 16000lb.$ the weight (W) that a power of 40*lb.* at the handle would sustain, supposing no friction. This however, is always very considerable; but less on square than on sharp threads.

From the two preceding examples it is easy to perceive that screws may be made to act with prodigious force. The Instrument called a Vice or *Vis* probably was distinguished by that name on account of its great power.

Respecting the *Inclined Plane*, it may be sufficient to refer to articles 343, 346, and 351.

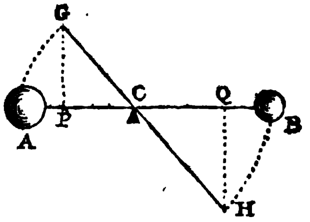
OF THE CENTER OF GRAVITY.

371. THE center of gravity is that point by which if a body be suspended, it shall hang or rest in any position.

Thus the center of a globe, and the middle of the axis of a cylinder, are their centers of gravity, if the bodies are uniformly dense.

Also the center of a circle, and the intersection of the diagonals of a parallelogram, are the centers of gravity. In speaking of the center of gravity of a surface however, we suppose that surface to be an indefinitely thin uniform lamina of matter.

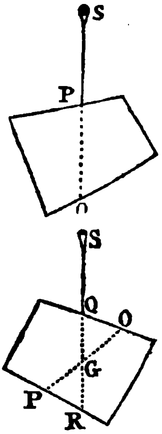
372. *Suppose the centers of the globes* A *and* B *are connected by an inflexible horizontal line or lever* AB *(without weight), and let* C *be the fulcrum or support; then if* $CA \times A = CB \times B$, *the bodies will be in equilibrio* (359); *and the fulcrum* C *is their center of gravity.*



For suppose the lever with the bodies to be turned on the fulcrum C into any other position GH; and let fall the perpendiculars GP, HQ. Then since the bodies are supported at the centers A and B, they gravitate or act on those points or the ends of the lever only, but in the directions GP, and QH when the lever is in the position GH. Now the triangles, CPG, CQH being similar, it follows that CP and CQ have the same ratio as CA and CB; hence $CP \times A = CQ \times B$, therefore the equilibrium still remains; and the bodies would rest in any position of the lever, provided it was always supported at the point C.

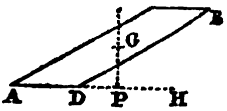
Corol. 1. Hence if the fulcrum or point of suspension were shifted towards B (for example), it is manifest the other body A would preponderate, and the lever turn till it rested in a vertical position, and consequently the centre of gravity (C) of the bodies would, in that case, be below the point of suspension, but in the same vertical line.

Corol. 2. Hence also, if a body be suspended at any point about which it can move freely by its own weight, the center of gravity will rest in the vertical line passing through the point of suspension. This suggests a method of finding mechanically the center of gravity of a thin flat body, thus,



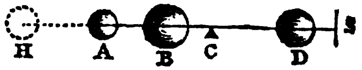
Suspend the body by a string SP at any point P on the edge, and mark the vertical PO on the surface by means of a plumb-line SO. Hang it up again by some other point Q, and draw the vertical QR; then because the center of gravity is in the line PO, and also in QR, it must be at their intersection G, or rather within the surface opposite that point.

And it follows that the body would remain at rest on a prop at R were the string SQ removed; but a very small weight or force when applied on either side of the vertical line QR would make it fall.



Corol. 3. Therefore when the vertical line (GP) passing through the center of gravity (G) of a body (AB) falls without the base (AD), the body will not stand on an horizontal plane (AH), because there is no part of the support or base in that vertical line; and consequently the body must turn on the base, and fall towards H. This is also the reason why a globe will not rest on a plane except it be horizontal.

373. *If three bodies,* A, B, D *of known weights, are connected by means of an inflexible line or lever* AD *passing through their centers of gravity, and the distances* AB, BD, *from each other are given; to find the common center of gravity* (C) *of the whole.*



This is the same thing as determining the fulcrum C when they rest in equilibrio. Therefore (359, corol. 4) it will be $CB \times B + CA \times A = CD \times D$.

Let the distance $AB = a$, $BD = b$, and $CD = x$; then $CB = b - x$; and we have $(b-x)\,B + (b-x+a)\,A = Dx$, which reduced

gives $x = \dfrac{Aa + Ab + Bb}{A + B + D} = CD$, the distance of the center of gravity C from the center of the body D.

Let S be a point in the line or lever AD produced; and put $DS = d$;

Then

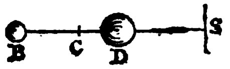
$$CS = \frac{Aa + Ab + Bb}{A + B + D} + d = \frac{Aa + Ab + Bb + Ad + Bd + Dd}{A + B + D};$$

$$\text{or } CS = \frac{(a + b + d)A + (b + d)B + dD}{A + B + D} = \frac{AS.A + BS.B + DS.D}{A + B + D}:$$

That is, if we consider S as a fulcrum or point of suspension, the sum of the forces or products $DS.D + BS.B +$ &c. divided by the sum of the bodies $D + B +$ &c. gives the distance of the common center of gravity of all the bodies from that point.

Corol. 1. Hence the center of gravity of any number of bodies in the same right line is readily determined:

For $\dfrac{HS.H + AS.A + BS.B + DS.D}{H + A + B + D}$ is the distance of the centre of gravity of the bodies H, A, B, D, from S; therefor making $DS = o$, we have $DS.D = o$, $HS = HD$ &c. and the expression becomes $\frac{HD.H + AD.A + BD.B}{H + A + B + D}$, for the distance from the body D.



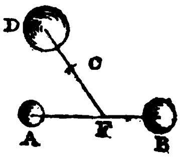
Corol. 2. And if C be the center of gravity of the bodies B and D; then $(B + D)\ CS = BS.B + DS.D$:

$$\text{For } \frac{BD.B}{B+D} = CD, \text{ and } CS = \frac{BD.B}{B+D} + DS;$$

$$\text{and } (B+D)\left(\frac{BD.B}{B+D} + DS\right) = (BD+DS)\,B + DS.D;$$

That is $(B+D)\ CS = BS.B + DS.D$.

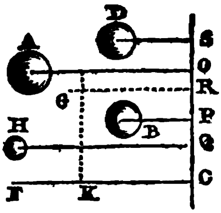
374. *The centers of gravity of three bodies* A, B, D, *any how situated, being given; to find their common center of gravity* C.



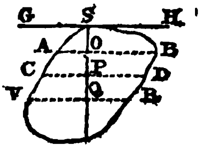
Join the centers of gravity of any two of them, suppose A and B: divide the distance AB reciprocally as the weights, that is take FB (for example) so that $A+B : AB :: A : BF$; then F is the center of gravity of the two bodies, or the place of the fulcrum upon which they would rest in equilibrio.

We now may consider F as the place of both bodies A and B, because that point sustains, or is pressed by their weight. Hence if DF be divided into two parts CD and CF having the ratio of $A+B$ to D, the point C will be the center of gravity of the three bodies. And in this manner, by taking two at a time, &c. the center of gravity of any number, or system of bodies, may be found.

375. *But the center of gravity of several bodies* A, B, D, H, *not situated in the same plane, may be determined thus,*



Let DS, AO, BP, HQ, drawn from the centers of gravity of the bodies, be perpendicular to a plane passing through SC. Then since all the points of suspension are in that plane, if we proceed with the distances as in Art. 373, we shall have $\frac{HQ.H+AO.A+BP.B+DS.D}{H+A+B+D}$ for the distance of the center of gravity from the plane. Suppose a plane GK parallel to SC to be at that distance from SC; and let the distance of the center be found from another plane CF, then the point which is the required center will be somewhere in the intersection of these two planes; and therefore a third plane, found in like manner, if it cuts that intersection, will determine the point.



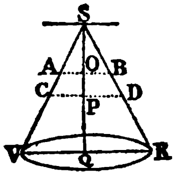
Corol. Hence if a body be suspended at a point S in the plane GH, and, after the manner of indivisibles, we suppose it to be composed of innumerable sections, AB, CD, VR, &c. parallel to the plane GH; then if SQ be perpendicular to GH,

$$\text{the sum of} \left\{ \begin{array}{l} SO \times \textit{section } AB \\ SP \times \textit{section } CD \\ SQ \times \textit{section } VR \\ \quad\&c. \qquad\qquad \&c. \end{array} \right\} \text{divided by the body}$$

will be the distance of the center of gravity from the plane GH. For if the body be homogeneous, the magnitude of the whole, or any part, is proportional to its weight.

Therefore if $d =$ the distance SO, or SP, &c. $s =$ the section AB, or CD, &c. and $B =$ the body,

Then $\frac{\text{all the } ds}{B} =$ the distance of the center of gravity from the plane GH. And by finding two other planes in which the center lies, its exact situation will be determined.



Example. Let SVR be a right cone suspended at the vertex. Then since the center of gravity of all the sections parallel to the base are in the axis SQ, the center of gravity of the cone must also be in that line.

Put the diameter of the base VR $=b$, the diameter CD of the section indefinitely near the base $=a$, the diameter AB of the next section $=c$, &c. $h=$ the height or axis QS, and $m=.7854$.

$$\text{Then } b : h :: a : \frac{ah}{b} = \text{SP},$$

$$b : h :: c : \frac{ch}{b} = \text{SO},$$

$$\&\text{c.} \qquad \&\text{c.}$$

$$\left.\begin{array}{ll} \text{And } mb^2 \times h & \text{or } b^3 \times \dfrac{mh}{b} \\ ma^2 \times \dfrac{ah}{b} & \text{or } a^3 \times \dfrac{mh}{b} \\ mc^2 \times \dfrac{ch}{b} & \text{or } c^3 \times \dfrac{mh}{b} \\ \&\text{c.} & \&\text{c.} \end{array}\right\} \begin{array}{l} \text{are the sections multiplied} \\ \text{by the distances from the} \\ \text{point of suspension:} \end{array}$$

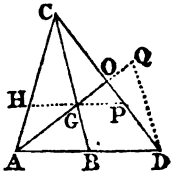
That is $(b^3+a^3+c^3+\&\text{c.}) \times \frac{mh}{b} =$ all the *ds*.

But since the indefinitely small distances QP, PO, &c. are supposed to be equal, the differences of VR, CB, AB, &c. will also be equal; and therefore $b^3+a^3+c^3+\&\text{c.}$ constitute an infinite series of cubes whose roots are in arithmetical progression; the greatest cube being b^3, and the least, or that at the vertex $=o$.

Now the sum of such a series is $\frac{n^4}{4}$, where n denotes the number of terms (179); and substituting b for n, we have $\frac{b^4}{4}$ for the sum of $b^3+a^3+c^3+$ &c. *in infin.* Therefore $\frac{b^4}{4} \times \frac{mh}{b}$ or $\frac{mhb^3}{4} =$ all the *ds*; this divided by $\frac{mb^2h}{3}$ (or B) the content of the cone, gives $\frac{3}{4}h$, the distance of the center of gravity from the vertex S.

Corol: Hence the center of gravity of an upright pyramid is also $\frac{3}{4}$ of its axis distant from the vertex.

To find the center of gravity of a Triangle (ACD).



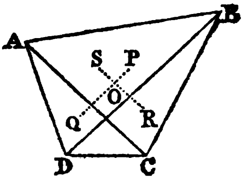
376. Bisect any two of its sides AD, CD, by lines CB, AO, drawn from the opposite angles; and the point of intersection G is the center of gravity.

Conceive the triangle to be composed of an indefinite number of right lines AD, HP, &c. parallel to the side AD; then since CB bisects those lines, or passes through their centers of gravity, the common center of gravity of the whole, or the centre of gravity of the triangle, must also lie in that line. And in the same manner it is proved that it falls in the bisecting line AO; consequently it must be at the intersection G.

If DQ be parallel to BC, the triangles OQD, OGC are equiangular, and because CO$=$OD, they are also equal, therefore DQ$=$CG; and since AB$=$BD, it follows, from similar triangles, that GB$=\frac{1}{2}$QD$=\frac{1}{2}$GC, therefore CB is trisected in G, and consequently CG$=\frac{2}{3}$CB. In like manner AG$=\frac{2}{3}$AO.

Corol. If CAD be the base of an upright prism, the center of gravity of the prism will be in the middle of the line drawn perpendicular to the base at the point G.

To find the center of gravity of a Trapezium (ABCD).



377. Draw the diagonals AC, DB; and find Q, P, the centers of gravity of the triangles ADC, ABC; and R, S, those of the triangles DCB, DAB; join QP, and RS: then, as the center of gravity of the trapezium lies in QP, and also in RS, its situation must therefore be at O the intersection of those lines.

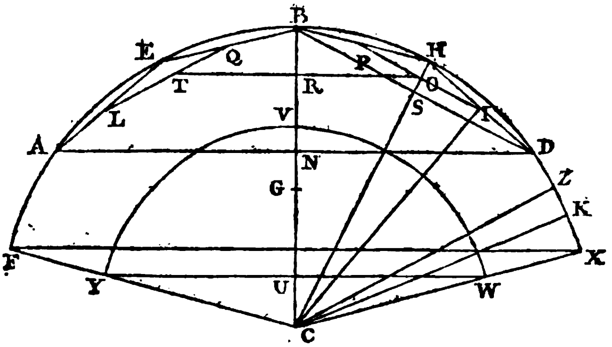
Or the center O may be found by dividing QP reciprocally as the areas of the triangles ACB, ACD; that is,

triang. ACB : *triang*. ACD :: OQ : OP.

To find the center of gravity of a Polygon.

378. The center of gravity of any regular polygon is evidently that of its inscribed or circumscribing circles. But if the polygon be irregular, divide it into triangles, and find their centers of gravity; then if we consider the magnitude of each triangle to be a weight placed at its center of gravity, the common center of gravity of the whole may be found by proceeding as in art. 374.

379. *If sides of a regular polygon* (AE, EB, BH, HD, &c.) *be inscribed in the segment of a circle; then, as half the sum of the sides* (DH+HB), *is to* (CI) *their distance from the center of the circle, so is* (ND) *half the chord of the segment, to* (CR) *the distance of the center of gravity of those polygonal sides from the center of the circle.*



Let the sides be bisected in L, Q, P, I; then if LQ and IP are also bisected in T and O, the intersection of TO and CB, or the point R, will evidently be the center of gravity of the chords or polygonal sides AE, EB, BH, HB.

The triangles DSH, COI are similar,

whence $CO : CI :: DS : DH$,

$:: 2DS\ (= BD) : 2DH\ (= DH + HB)$;

or $CO : BD :: CI : DH + HB$.

And from the similar triangles ORC, BND,
we get CO : BD :: CR : ND;
Therefore by equality $DH+HB$: CI :: ND : CR.

Corol. 1. If we suppose the sides of the polygon to be diminished indefinitely, so as to coincide with the arc, then half the sum of the sides is equal to half the arc, and CI the perpendicular becomes equal to the radius; hence

As half any arc of a circle, to half its chord, so is the radius of the circle, to the distance of the center of gravity of the arc from the center of the circle.

Corol. 2. Conceive the sector FBXC to be divided into an infinite number of triangles CXK, CKZ, &c. the bases XK, KZ, &c. being considered as right lines; then (376) their centers of gravity will be $\frac{2}{3}$ CX distant from the center C. Let $CW = \frac{2}{3} CX$, and describe the arc WVY; then the centers of gravity of all the triangular spaces will be in that arc; consequently the center of gravity of the arc will also be that of all the triangles, or of the sector FBCX; therefore if G be the center of gravity of the arc WVY, or of the sector,

arc VW : UW (half its chord) :: CW : CG.

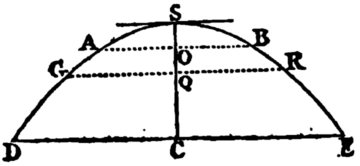
But the sectors BCX, VCW are similar, and $CW = \frac{2}{3} CX$,
whence, *arc* BX : $\frac{1}{2}$ *chord* FX :: $\frac{2}{3}$CX : CG,
or, as the *arc* FBX : *chord* FX :: $\frac{2}{3}$ *radius* CX : CG.

Corol. 3. If t and x respectively denote the distances of the centers of gravity of the triangle CFX, and the circular segment FBX, from G,

Then $triang. \times t = segm. \times x$, whence $\frac{triang. \times t}{segm.} = x$ the distance of the centre of gravity of the segment from that of the sector.

To find the center of gravity of a Parabola.

380. It is manifest the center of gravity lies in the axis.



Let the parabola be suspended at its vertex S: and put $a =$ SC the axis, and $p =$ the parameter. Also suppose the surface to be composed of an infinite number of lines AB, GR, &c. parallel to DE, and at equal distances SO, OQ, &c. from one another:

Then, from the nature of the parabola,

$$2\sqrt{p\mathrm{SO}} = \mathrm{AB},$$
$$2\sqrt{p\mathrm{SQ}} = \mathrm{GR},$$
&c. &c.

And (375. corol.) $2\sqrt{p\mathrm{SO}} \times \mathrm{SO} + 2\sqrt{p\mathrm{SQ}} \times \mathrm{SQ} + \&c. \ldots\ldots 2\sqrt{p\mathrm{SC}} \times \mathrm{SC}$,

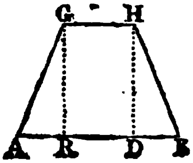
or $2\sqrt{p\mathrm{SO}^3} + 2\sqrt{p\mathrm{SQ}^3} + \&c. \ldots\ldots\ldots\ldots 2\sqrt{pa^3}$

That is $(\mathrm{SO}^{\frac{3}{2}} + \mathrm{SQ}^{\frac{3}{2}} + \&c. \ldots\ldots\ldots a^{\frac{3}{2}})\, 2\sqrt{p} =$ all the *ds*.

Now, by Art. 179, the sum of the series $\mathrm{SO}^{\frac{3}{2}} + \mathrm{SQ}^{\frac{3}{2}} + \&c. \ldots\ldots\ldots\ldots a^{\frac{3}{2}}$ (that is, from $o^{\frac{3}{2}}$ at S, to $a^{\frac{3}{2}}$ at C) will be $\frac{a^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{a^{2\frac{1}{2}}}{2\frac{1}{2}}$; therefore $\frac{a^{\frac{5}{2}}}{2\frac{1}{2}} \times 2\sqrt{p} =$ all the *ds*; which divided by $\frac{4}{3}\sqrt{pa^3}$ the area of the parabola, (or B), gives $\frac{\frac{2}{5}\sqrt{a^5}}{\frac{2}{3}\sqrt{a^3}} = \frac{3}{5}a$, the distance of the center of gravity from the vertex S.

To find the center of gravity of the frustum of a square Pyramid.

381. Let $l =$ GH the side of the less end, $g =$ AB the side of the base or greater end, and $h =$ the height of the frustum.



If the frustum be cut through the 4 sides of the less end by planes perpendicular to the base, it will be divided into 4 equal square pyramids, 4 equal triangular prisms, and a parallelopiped GHDR, all of the same altitude. Each pyramid having AR or DB or $\frac{g-l}{2}$ for the side of its base; and AR or DB, and GH or RD or their equals $\frac{g-l}{2}$ and l, are the sides of the bases of the prisms *.

Now $\left(\frac{g-l}{2}\right)^2 \times \frac{1}{3}h \times 4 = (g^2-2gl+l^2)\frac{1}{3}h =$ the cubic contents of the 4 pyramids.

$\frac{g-l}{2} \times \frac{1}{2}h \times l \times 4 = (gl-l^2)h$............of the 4 prisms.

$l^2 \times h$.........of the parallelopiped.

And the aggregate is $(g^2+gl+l^2)\frac{1}{3}h$, the content of the frustum.

Suppose the frustum to be suspended at the least end GH;

Then $\frac{3}{4}h$ is the distance of the center of gravity of the pyramids
$\frac{2}{3}h$ .... of the prisms
$\frac{1}{2}h$ .... of the parallelopiped
} from that end.

And $\frac{(g^2-2gl+l^2)\frac{1}{3}h \times \frac{3}{4}h + (gl-l^2)h \times \frac{2}{3}h + l^2 \times h \times \frac{1}{2}h}{(g^2+gl+l^2)\frac{1}{3}h}$ will be, the distance in the axis, of the center of gravity of the frustum from the least end, or plane of suspension GH, (375). This expression reduced becomes $\frac{3g^2+2gl+l^2}{g^2+gl+l^2} \times \frac{1}{4}h$,

Corol. And the same expression will answer for the frustum of a cone or any upright pyramid, if g and l denote the diameters, or other similar lines of the greater and less ends; because the surfaces will be as the squares of those diameters, or lines,

Example. Let the frustum be a squared piece of timber 30 *feet* long, and the sides of the greater, and less ends = 2, and $1\frac{1}{2}$ *feet*, respectively;

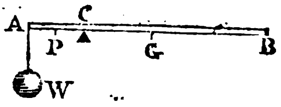
Then $\frac{12+6+2\frac{1}{4}}{4+3+2\frac{1}{4}} \times \frac{30}{4} = 16\frac{31}{74}$ *feet*, the distance of its center of gravity from the least end,

382. *Suppose* AB *is a squared beam, or lever of oak,* 30 *feet long, each end being a foot square; now what weight* W *at the end* A, *would keep it in an horizontal position, on a fulcrum* C, 3 *feet from that end, if each cubic foot of the beam weighs* 57lb.?

---

* These sections will readily be comprehended if the exterior lines are drawn on a model, Which may be cut from a common cork, or soft wood.

Since 1 foot in length is also a cubic foot, we have $27 \times 57 = 1539$*lb.* the weight of the arm CB, and $3 \times 57 = 171$*lb.* that of CA.



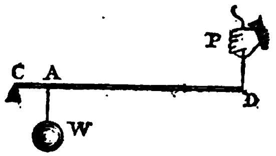
CG $= 13\frac{1}{2}$, and CP $= 1\frac{1}{2}$ feet, the distances of the centers of gravity of the arms from the fulcrum or prop C.

We may now consider a weight of 1539*lb.* at G, another of 171*lb.* at P, and a third at A, all suspended on a lever void of gravity, and resting in equilibrio on the support C: and we have

$$W \times CA + P \times CP = G \times CG, \text{ (359 corol. 4)}$$

whence $W = \dfrac{G \times CG - P \times CP}{CA} = \dfrac{1539 \times 13\frac{1}{2} - 171 \times 1\frac{1}{2}}{3} = 6840$*lb.* the weight or force required.

383. *If* CD *be an uniform iron bar, or lever of the second kind,* 6 *feet in length, weighing* 36lb. *it is required to find what power* P *would be sufficient to sustain the lever, and weight* W = 64lb. *in equilibrio, if the distance* CA = 16 *inches?*



The lever being supposed a right line, we have $DC \times P = CA \times W$ (360), whence $P = \dfrac{CA \times W}{CD}$, which, in that case, would be the power; but half the weight of the lever, or $\frac{1}{4}$CD, is sustained by the power P, (CD being inches); therefore P is $= \frac{1}{4}CD + \dfrac{CA \times W}{CD} = 18 + \dfrac{16 \times 64}{72} = 32\frac{2}{9}$*lb.* the power required.

Scholium. Were the lever without gravity, it is manifest, by increasing its length, the necessary power would be diminished. But when the lever is an iron bar, beam of wood, &c. there is a certain determinable length which admits of a *minimum* power that will sustain, or raise a given weight (W) at a given distance (CA) from the prop C. Thus in the present example, let the length CD $= x$ inches; then $\frac{1}{4}x =$ half the weight of the lever in pounds; and $\frac{1}{4}x + \dfrac{CA \times W}{x} =$ the power P.

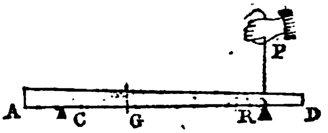
Suppose $\frac{1}{4}x + \dfrac{CA \times W}{x} = m$, then by reduction we have the quadratic equation $x^2 - 4mx = -4CA \times W$;

$$\text{whence } x = 2m \pm \surd\,(4m - 4CA \times W):$$

Now the least possible value of m is when $4m^2 - 4CA \times W = o$, or $4m^2 = 4CA \times W$; for if $4m^2$ be *less* than $4CA \times W$, the expression $\sqrt{(4m^2 - 4CA \times W)}$ becomes impossible. Hence, when $4m^2 = 4CA \times W$, we get $m = \sqrt{(CA \times W)}$; and $x = 2m \pm o = 2\sqrt{(CA \times W)} = 2\sqrt{(16 \times 64)} = 64$ *inches*, the length of the lever,

And $P = \frac{64}{4} + \frac{16 \times 64}{64} = 32$lb. The least power by which the weight can be sustained in equilibrio, when the lever weighs half a pound per inch.

384. *A piece of timber nearly in the shape of a conic frustum, 40 feet long, is supported in an horizontal position on two props* C *and* R, 6 *feet from the ends; now if the diameter of the greater end be* 2 *feet, and that of the less* 1 *foot, what is the pressure on each prop?*



Let g and l denote the diameters of the ends A and D, and h the length,

Then (381) $\frac{12+4+1}{4+2+1} \times 10 = 24\frac{2}{7}$ feet, $=$ DG the distance of the center of gravity G from the less end; therefore $GC = 9\frac{5}{7}$, and $GR = 18\frac{2}{7}$.

If $s =$ the weight, or the cubic contents of the frustum, we may conceive that weight, or contents, to be suspended at the center of gravity G on a lever AB void of gravity, and supported in equilibrio on the fulcrum C by a power P instead of the prop R;

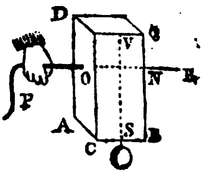
Then (360) $CG \times s = CR \times P$; whence $\frac{CG \times s}{CR} = P$;

or 28 (CR) : $9\frac{5}{7}$ (CG) :: s : P, the pressure on R;

and 28 : $18\frac{2}{7}$ (GR) :: s : the pressure on C:

That is, the whole weight must be divided reciprocally as the distances of the props from the center of gravity of the body.

385. *Let* DB *be a heavy body in the form of a parallelopiped, standing on the base* ACB *perpendicular to the horizon; to find what power* P *acting parallel to the horizon, at a given height* CO *or* BN *above the base, would be sufficient to turn it over.*



Let the perpendicular VS bisect CB; then as the center of gravity of the rectangle CQ is in that perpendicular, we may consider the surface CQ as being collected into a weight suspended at the point S directly under the center of gravity. And if CB were at liberty to move about the point C, the force at S to turn it over is SC × the weight at S, or SC × surface CQ, therefore considering SCO as a bended lever, and C the fulcrum, we shall have SC × surface CQ = CO × power at P, in the case of an equilibrium; hence $\frac{SC \times \textit{surface } CQ}{CO} = P$.

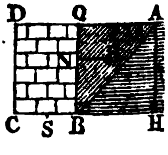
Now the body is composed of innumerable planes parallel and equal to CQ, therefore substituting w the weight of the body or of all those planes, for CQ, we have $\frac{SC \times w}{CO} = P$, or $CO : SC :: w : P$, the force necessary to keep all the planes or the body in equilibrio on the edge or line CA, provided it rested on that edge only.

Let CB = CA = 3 *feet*, the height BQ = 6 *feet*, CO or BN = 4 *feet*, and suppose the body to be heavy stone or marble, weighing 160*lb.* per cubic foot. Then, $3 \times 3 \times 6 \times 160 = 8640lb. = w$ the weight:

$$\text{And } \frac{SC \times w}{CO} = \frac{1\frac{1}{2} \times 8640}{4} = 3240lb.$$

Therefore it would require a force something greater than 3240*lb.* when applied at the height of 4 feet, either to pull or push it down.

386. *Let* BQAH *be the perpendicular section of a bank of earth; to find the thickness* CB *of an upright rectangular wall necessary to support it.*



If the bank consisted of loose earth without any support on the side QB, a part QBA would slide down, leaving the slope AB inclined to the horizon CH in a greater, or less angle, according as the earth was more, or less tenacious. Sand and fine gravel will descend till the angle ABQ is less than 30°; but a slope greater than 60° may be formed with some stiff soils. On these accounts, the inclination of AB is usually taken at 45° in computations, as a sort of medium.

Let $h = \text{BQ}$ the height of the wall, and $x = \text{BC}$ its thickness; then if $\text{QA} = \text{QB}$, $\frac{1}{2}h^2$ is the area of the triangle QBA, and $hx =$ that of the rectangle DB or section of the wall.

Now if we consider the triangle QBA as a body at liberty to descend down the plane AB without friction, its force against QB in an horizontal direction RN will be equal to its weight $\frac{1}{2}h^2$ (denoting its weight by the area or surface): for it is sustained in equilibrio by the resistance of BQ, which resistance is perpendicular to BQ: therefore (344, corol. 2) BH (or QA) : BQ :: *weight* $\frac{1}{2}h^2 : \frac{1}{2}h^2$, the force in the horizontal line NR; R being the center of gravity of the triangle.

Let hx be considered as a weight at S the middle of BC (as in the preceding article); then C being the fulcrum of the bended lever NBC, and $\text{BN} = \frac{2}{3}\text{BQ}$ (376),

$$\text{we have CS} \times hx = \text{BN} \times \tfrac{1}{2}h^2, \text{ in the case of an equilibrium;}$$
$$\text{or } \tfrac{1}{2}x \times hx = \tfrac{2}{3}h \times \tfrac{1}{2}h^2; \text{ whence } x = h\sqrt{\tfrac{2}{3}},$$
$$\text{or BC} = .816\,\text{BQ}:$$

That is, when the wall is built with materials of the same weight as the earth, its thickness must exceed $\frac{8}{10}$ of the height. This is according to the example in Dr. Hutton's Course of Math. vol. II.

Muller however, (*Practical Fortification*) by allowing $\frac{2}{3}$ of the pressure for friction on the plane AB, reduces the force of the triangle to $\frac{1}{3} \times \frac{1}{2}h^2$ or $\frac{1}{6}h^2$ against the point N;

$$\text{Hence } \tfrac{1}{2}x \times hx = \tfrac{2}{3}h \times \tfrac{1}{6}h^2; \text{ and } x = h\sqrt{\tfrac{2}{9}} = .47h,$$
$$\text{or BC} = .47\,\text{BQ}:$$

That is, the thickness is nearly half the height.

But M. Belidor (*Science des Ingenieurs*) endeavours to prove that the triangle QAB should first be diminished to half its pressure or weight on account of the tenacity of the earth. He then considers the parts of the triangle as acting separately against QB in directions parallel to the slope AB, and reduces all their forces to the point Q. The same conclusion nearly however, is obtained by taking $\frac{2}{7}$ of the triangle, or $\frac{2}{7} \times \frac{1}{2}h^2$ for the force acting in an horizontal direction against the point N; and therefore we shall have

$$\tfrac{1}{2}x \times hx = \tfrac{2}{3}h \times \tfrac{4}{7} \times \tfrac{1}{4}h^2, \text{ whence } x = h\sqrt{\tfrac{4}{21}} = .436h;$$
$$\text{or } BC = .436\,BQ:$$

which is not greatly different from the conclusion according to Muller.

To compute the thickness when the wall or revetment is of brick, or of stone. Let $e = 124lb.$ the weight of a cubic foot of common earth; $b = 125lb.$ that of a cubic foot of brick; and $s = 158lb.$ the weight of stone per cubic foot. Then 124, 125, 158, or any three numbers in the same proportion, will denote their specific gravities. And since the weights of bodies are as their specific gravities, if the wall be of brick we shall have

$$\left.\begin{array}{l} \tfrac{1}{2}hx^2 \times b = \tfrac{1}{3}h^3 \times e, \text{ and } x = h\sqrt{\frac{2e}{3b}} = .813h = \frac{13}{16}h \\ \tfrac{1}{2}hx^2 \times b = \tfrac{1}{9}h^3 \times e \ldots\ldots\ldots x = h\sqrt{\frac{2e}{9b}} = .47h = \frac{8}{17}h \\ \tfrac{1}{2}hx^2 \times b = \tfrac{2}{21}h^3 \times e \ldots\ldots\ldots x = h\sqrt{\frac{4e}{21b}} = .43h = \frac{3}{7}h \end{array}\right\} \text{nearly,}$$

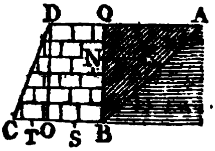
That is, if the wall be 16 *feet* high, its thickness, according to the first hypothesis, should be rather more than 13; but Belidor's makes it about 7 *feet*.

If the wall be of stone, then

$$x = h\sqrt{\frac{2e}{3s}} = .72h$$
$$x = h\sqrt{\frac{2e}{9s}} = .42h$$
$$x = h\sqrt{\frac{4e}{21s}} = .39h$$

Hence, supposing the height BQ = 16 *feet*, the different hypotheses give about $11\frac{1}{4}$, $6\frac{7}{10}$, and $6\frac{1}{5}$ *feet*, respectively, for its thickness.

387. *If* CDQB *be the profil of a revetment or wall supporting the earth* QBA; *to find the thickness* DQ *or* OB *when the slope* DC *is given.*



Let $h =$ OD or BQ, $nh =$ CO the base of the triangle CDO, $x =$ BO, also suppose e, b, s to denote the same specific gravities as in the last article.

Then $hx =$ the rectangle DB, and $\frac{1}{2}h^2 =$ the triangle QBA (as above); also $\frac{1}{2}nh^2 =$ the triangle CDO. Now instead of finding the center of gravity of the trapezoid CDQD, we shall conceive the surfaces of the rectangle DB, and the triangle CDO to be weights at S and T directly under their centers of gravity, S being the middle of OB (as before); but T will be $\frac{2}{3}$ CO distant from C; that is CT $= \frac{2}{3}nh$, and CS $= nh + \frac{1}{2}x$.

Then C being the fulcrum of the bended lever CBN, we have

$$\tfrac{2}{3}nh \times \tfrac{1}{2}nh^2b + (nh + \tfrac{1}{2}x)\,bhx = \tfrac{1}{3}h^3e,$$

or $\frac{1}{3}n^2h^3b + nbh^2x + \frac{1}{2}hbx^2 = \frac{1}{3}h^3e$ in the case of an equilibrium when the wall is of brick, according to the first hypothesis in the preceding article. This expression reduced

gives $x + nh = \sqrt{\dfrac{2h^2e + n^2h^2b}{3b}}$, and $x = h\sqrt{\left(\dfrac{2e}{3b} + \dfrac{n^2}{3}\right)} - nh$.

Suppose CO $= \frac{1}{3}$ of the height OD, that is, let $n = \frac{1}{3}$,

then $x \doteq h\sqrt{\left(\dfrac{2e}{3b} + \dfrac{n^2}{3}\right)} - nh = .62h$, nearly.

But adopting the second hypothesis (Muller's), it will be

$\frac{1}{3}n^2h^3b + nbh^2x + \frac{1}{2}hbx^2 = \frac{1}{9}h^3e$; whence $x = .28h$.

And taking $\frac{2}{21}h^3e$ instead of $\frac{1}{9}h^3e$, we get $x = h\sqrt{\left(\dfrac{4e}{21b} + \dfrac{n^2}{3}\right)} - nh = .25h$; or DQ $\doteq \frac{1}{4}$QB, according to Belidor.

If the revetment be of stone, then substituting s for b, we get

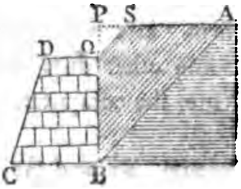
$$\left.\begin{aligned} x &\doteq h\sqrt{\left(\frac{2e}{3s} + \frac{n^2}{3}\right)} - nh = .53h \\ x &\doteq h\sqrt{\left(\frac{2e}{9s} + \frac{n^2}{3}\right)} - nh = .23h \\ x &= h\sqrt{\left(\frac{4e}{21s} + \frac{n^2}{3}\right)} - nh = .2h \end{aligned}\right\}\ \text{nearly.}$$

Suppose the height BQ $= 15$ feet,

$$\left.\begin{aligned} \text{then }\ .53 \times 15 &= 7 \text{ feet} \\ .23 \times 15 &= 3\tfrac{1}{2} \\ .2\ \times 15 &= 3 \end{aligned}\right\}\ \text{nearly, for DQ the thickness at top.}$$

And by adding CO $= 3$ feet ($\frac{1}{5}$ of BQ) we shall have CB the thickness at bottom.

388. *When the revetment supports a bank of earth* QBAS *raised above the top* DQ: Let T = the area of the triangle PBA, and t = that of the triangle PQS; also suppose r to denote 1, or $\frac{1}{3}$, or $\frac{2}{7}$.



Then $\frac{2}{3}\mathrm{BP} \times Tr =$ the force of the triangle PBA to turn the bended lever PBC about the fulcrum C; and $(\mathrm{BQ} + \frac{2}{3}\mathrm{QP}) \times tr =$ that of the triangle PQS;

and their difference $\frac{2}{3}\mathrm{BP} \times Tr - (\mathrm{BQ} + \frac{2}{3}\mathrm{QP}) \times tr$ is the effort of the quadrilateral QBAS.

Hence if the wall be of stone, we shall have

$\frac{1}{3}n^2h^3s + nsh^2x + \frac{1}{2}hsx^2 = \frac{2}{3}\mathrm{BP} \times Tre - (\mathrm{BQ} + \frac{2}{3}\mathrm{QP}) \times tre$, when the revetment and bank are in equilibrio.

Suppose $\mathrm{QP} = 9$ feet, and the talus QS parallel to BA, or the angle $\mathrm{PQS} = 45°$; the other dimensions remaining as in the last article:

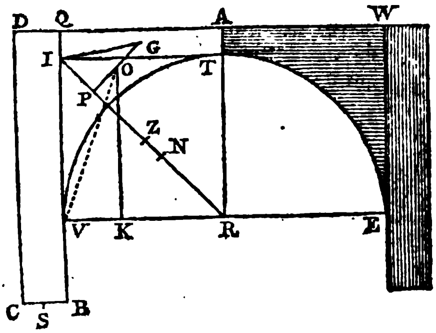
Then the equation reduced gives $x^2 + 6x + 6 = 401.56r$.

$$\left.\begin{array}{l} \text{If } r = 1, \text{ then } x = 17.1 \text{ feet,} \\ \quad r = \frac{1}{3} \;\ldots\ldots\; x = 8.6 \\ \quad r = \frac{2}{7} \;\ldots\ldots\; x = 7.8 \end{array}\right\}$$ nearly, the thickness QD, according to the different hypotheses.

Remark. If the triangle QBA, (*fig*. preceding the last), or rather the triangular prism of earth of which QBA is a perpendicular section, were a solid body on a perfectly smooth plane AB, its force in the horizontal direction RN would be denoted by $\frac{1}{2}\mathrm{BQ}^2$ at all inclinations of the slope AB. For $\frac{1}{2}\mathrm{BQ} \times \mathrm{QA}$ = the area of the triangle QBA; consequently (344, corol. 2) $\mathrm{QA} : \mathrm{QB} :: \frac{1}{2}\mathrm{QB} \times \mathrm{QA} : \frac{1}{2}\mathrm{BQ}^2$ its force in the direction RN, as in the first of the preceding methods. But this supposition gives the thickness of walls nearly double what M. Vauban found by experience to be necessary. Soils however, are so various, that we cannot expect any rule, even from experiments, which will give a satisfactory result in all cases.

In these computations, the wall is considered as one compact block, or the joinings as strong as the solid material; and that its resistance arises from the weight only. But if the wall be firmly attached to a foundation sunk in the earth, it is manifest some other *data* derived from the *strength* of the wall, must enter the computations.

389. *To find the thickness of the piers necessary to support a semicircular Arch* VQWETV.



Let VTAQ be a perpendicular section of half the arch, O its center of gravity; and CBQD the corresponding section of the pier supporting that half arch.

Now the arch is supposed to be of such materials, that were it not opposed by the pier DB, its own weight would break it at TA: the arch therefore exerts its force or weight in three directions, namely in the perpendicular direction OK, in that of OV, and in an horizontal direction KV; and the forces will be as those three lines: but OK, which is in the direction of gravity, must therefore be proportional to the weight. Hence if $w =$ the weight of the arch, or the surface VTAQ (to which it is proportional),

Then $OK : w :: KV : \frac{wKV}{OK} =$ the lateral pressure at V, or that in the horizontal direction KV.

We now consider C as the fulcrum of the bended lever CBV, and suppose a weight at S the middle of CB, equal to the surface DCBQ, or $= CB \times BQ$;

Then $VB \times \frac{wKV}{OK}$ is the effort of the arch at V in the direction KV to turn it on the point C, and $CS \times CB \times BQ$ that of DB on the same point C, in a perpendicular direction: consequently, in case of an equilibrium, those forces must be equal; that is,

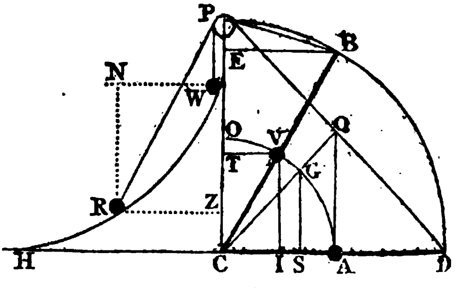
$$\tfrac{1}{2}CB \times CB \times BQ = VB \times \frac{wKV}{OK}; \text{ whence } \tfrac{1}{2}CB^2 = \frac{VB \times wKV}{BQ \times OK}.$$

Let the radius RV or RT = 30, TA = 4, and VB = 12 *feet*; then QB = 46. Now to find O the center of gravity of the surface VTAQ;

Let TI be parallel to AQ: draw IR and IG, G being the middle or center of gravity of the parallelogram IQAT. And if N be the center of gravity of the quadrant RTV, RN = 18 *feet* nearly (379 corol. 2). Also the area of the quadrant RTV = 706.86 *feet*, that of the square ITRV = 900, and the difference 193.14 = the area VTI; and since Z the middle of IR, is the center of gravity of the square ITRV, NZ is = 3.213; hence, if P be the center of gravity of the space ITV, we shall have (372) PZ $= \frac{706.86 \times 3.213}{193.14} = 11.76$, therefore IP = 9.453 nearly. Moreover, since GI is = $\frac{1}{2}$ the diagonal of the parallelogram QT, we get (by trigonom.) PG = 12.024, which being divided reciprocally as the two surfaces IQAT, and VTI, gives GO = 7.414, and O is the common center of gravity of both surfaces or of the quadrilinear space VTAQ or section of the half arch. And hence OK and KV are readily found to be 26.64, and 9.87 *feet*, respectively.

These values being substituted in the above expression, we shall have $\frac{1}{2}BC^2 = \frac{12 \times (193.14 + 120) \times 9.87}{46 \times 26.64}$, whence BC = 7.8 *feet*, nearly, the thickness, when the pier just prevents the arch from falling, consequently the dimensions should be somewhat greater. Also, when the pier stands in water, its pressure will be lessened, and an allowance ought to be made on that account, except it be supported on the side DC.

390. *Suppose* CD *is a beam of wood moveable about the end* C, *and supported by the weight* W *attached to a flexible line* DPW *passing over a pulley at* P; *to determine the curve* WRH, *along which the weight* W *must move, so that the beam and weight shall always be in equilibrio.*



Let CD be horizontal, and the perpendicular CP = CD; also suppose the beam is of uniform thickness; then we may consider it as a line or lever without gravity, having a weight at the middle point A (its center of gravity) equal to that of the beam. Make CQ perpendicular to PD, and QA will be perpendicular to CD.

Since CQ and CA are respectively perpendicular to DP and QA the directions in which the weights W and A act on the lever to turn it about the end C, the weights or forces in equilibrio, will be reciprocally as those perpendiculars CQ and CA, (by the properties of the lever);

that is, CQ : CA :: *weight* A : *weight* W;

or CG : CS :: *weight* A : *weight* W, (by sim. triang.).

Now suppose CB to be another position of the beam or lever, and R the corresponding place of the weight W: Draw RZ, VT, BE parallel to CD, and VI parallel to PC; then WZ is the perpendicular descent of the weight W, and IV the corresponding vertical ascent of the weight A, and those spaces are reciprocally as the weights or forces, in the case of an equilibrium, (343);

That is, WZ : IV :: *weight* A : *weight* W :: CG : CS (by equality):

Hence it appears that WC = CQ : for WZ : IV : CG : CS :: CQ : CA or CO; that is, WZ : IV :: CQ : CO; but when the beam is vertical or in the position CP, V and O coincide, and IV becomes = CO, therefore the antecedents WZ and CQ are also equal.

Let WZ $= x$, ZR $= y$, CD or CP $= h$, PW $= p$, OT $= v$, $s =$ the length of the line DPW, $m =$ CG, $n =$ CS or GS:

Then WZ : IV :: CG : CS (or CQ : CO);

that is, $x : \frac{1}{2}h - v :: m : n$; whence $v = \dfrac{\frac{1}{2}mh - nx}{m}$; but the sectors OCV, PCB are similar, and PC $=$ 2OC, whence $2\text{OT} = 2v = \dfrac{mh - 2nx}{m}$.

And because the triangle PCB is isoceles, $\text{PB}^2 = 2\text{PC} \times \text{PE}$ $= 2h \times \dfrac{mh - 2nx}{m} = \dfrac{2mh^2 - 4nhx}{m}$, therefore $\text{PB} = \sqrt{\dfrac{2mh^2 - 4nhx}{m}}$;

consequently $s - \sqrt{\dfrac{2mh^2 - 4nhx}{m}} = \text{PR}$; and PZ $= p + x$, whence $\left(s - \sqrt{\frac{2mh^2 - 4nhx}{m}}\right)^2 - (p + x)^2 = y^2$ $(= RZ^2)$, the equation of the curve, exhibiting the relation of an ordinate WZ or NR (x), to its corresponding abscisse ZR or WN (y).

Suppose CD = CP = 20 *feet*, its weight or the weight of A = 600*lb*. then CS being the side of a square, and CG its diagonal, m and n may be denoted by 1.414 &c. and 1; and we have

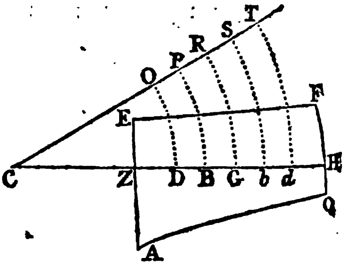
1.414 : 1 :: 600 : 424*lb*. the weight W.

Also $\sqrt{200} = 14.14$ &c. *feet* = CQ = CW; and PW = 5.86; whence s = DPW = 34.14 *feet*; and CH = $\sqrt{(34.14^2 - 20^2)} = 27.67$ *feet*, nearly.

If we assume x, and find the corresponding values of y, the curve may be traced by means of points: Thus, suppose WZ or $x = 10$, which being substituted for x in the equation of the curve, gives ZR or $y = 10.16$ *feet*, nearly.

If CV be the radius, the perpendicular ascent of the weight A will always be denoted by (VI) the sine of the inclination of CB to the horizon; hence M. Belidor calls this curve the *Sinusoid:* see his *Science des Ingenieurs*, where a weight (W) moveable along the curve, is made the counterpoise to a Draw-bridge (CD) that turns on the end C.

391. *Let the plane* AEFQ *be perpendicular to the horizon, G its center of gravity, and* CGH *parallel to the horizon; then if the plane revolves about* C *as a center, always retaining its vertical position, the solid it generates, is equal to the said plane drawn into the arc* (GR) *described by its center of gravity.*

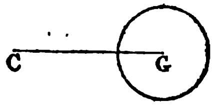


Conceive the whole surface AEFQ to be collected in, or reduced to the axis or line ZH by an indefinite number of perpendiculars to that line drawn through the surface; and suppose the distances GB, Gb; GD, Gd, &c. are reciprocally as the number of particles in the points B, b, D, d,

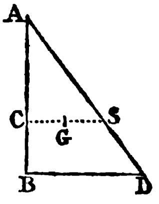
&c. respectively; then if B, b, D, d, &c. denote the number of particles in those points, we have (by prop. of the lever) $GB.B = Gb.b$, $GD.D = Gd.d$, &c. and therefore (373, corol. 2) $CB.B + Cb.b = CG\ (B+b)$, $CD.D + Cd.d = CG\ (D+d)$, &c. But the arcs DO, BP, &c. are respectively as the radii CD, CB, &c. hence, by taking those arcs instead of their radii, we get $BP.B + bS.b = GR\ (B+b)$, $DO.D + dT.d = GR\ (D+d)$, &c.; whence $BP.B + bS.b + DO.D + dT.d$, &c. $= GR\ (B+b+D+d$, &c.): now the whole solid is made up of all the $BP.B$, $bS.b$, &c. taken together, therefore $GR\ (B+b+$ &c.), or $GR \times$ *surface* AEFQ is the solid. And the like is also true of surfaces described by lines.

Examples.

1. If the circle whose center is G, revolve about C, it will generate a ring (like the ring of an anchor); its solid content will therefore be = the surface of the circle multiplied by the space described by the center of gravity G, or the circumference of the circle whose radius is CG. This is also known from other principles.

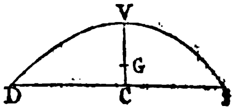


2. To find the content of a cone, or the solid generated by the revolution of a right angled triangle ABD about the perpendicular BA.



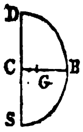
If $AC = \frac{2}{3}AB$, and CS parallel to B, then G, the middle of CS, is the center of gravity of the triangle; that is, $CG = \frac{1}{3}BD$. And the circle described by the point G in one revolution $= \frac{2}{3}BDc$ (c being $= 3.1416$), this multiplied by $\frac{1}{2}AB \times BD$ (the area of the triangle), is $\frac{2}{3}BDc \, . \, \frac{1}{2}AB \, . \, BD = BD^2c \, . \, \frac{1}{3}AB$; but BD^2c is the area of the base of the cone, or of the circle described by BD; therefore the base multiplied by $\frac{1}{3}$ of the height gives the solid content.

3. To find the content of a parabolic spindle, or the solid generated by the revolution of a parabola DVS about an ordinate DS.



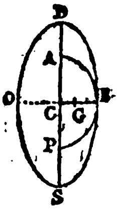
If G be the center of gravity of the parabola, $GC = \frac{2}{5}CV$ (380); hence, putting $c = 3.1416$, we shall have $\frac{4}{5}CVc =$ the circumference of the circle described by the point G. And since $\frac{2}{3}DS . CV$ is the area of the parabola (304) or the generating surface, we get $\frac{4}{5}CVc . \frac{2}{3}DS . CV = \frac{8}{15}CV^2c . DS$: but $CV^2c . DS$ is the content of the circumscribing cylinder. Therefore a parabolic spindle is $\frac{8}{15}$ of its circumscribing cylinder, when its axis (DS) is at right angles to (CV) the axis of the generating parabola.

4. Let it be required to find the surface of a sphere described by the revolution of a semicircular arc DBS about the diameter DS.



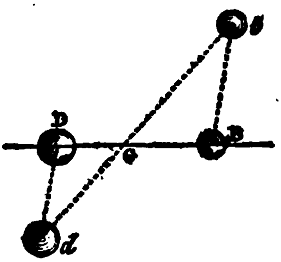
Suppose C to be the center of the circle, G the center of gravity of the arc DBS; then (379, corol. 1) $\frac{1}{2}DBS : CS :: CS : \frac{CS^2}{\frac{1}{2}DBS} = CG$; and (putting $c = 3.1416$), $\frac{4CS^2c}{DBS}$ is the track or circumference described by the center of gravity G in one revolution; this multiplied by DBS the generating line, gives $4CS^2c$; that is, $2CS$ the diameter of the sphere, multiplied by $2CSc$ its circumference, gives the superficies.

5. By a reverse operation, the center of gravity of a given surface may sometimes be determined. Thus, let it be proposed to find the center of gravity G of the semiellipse DBSCD, DS being the transverse axe, and CB the semiconjugate.



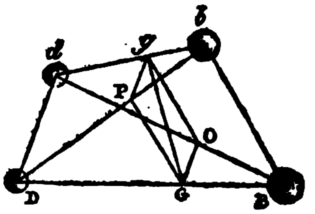
Put $t = CD$ the semitransverse, $g = CB$ the semiconjugate, $c = 3.1416$, and $x = CG$. Then (274. corol. 3) $\frac{1}{2}tgc$ is the area of the semiellipse. And $2cx =$ the circumference described by the center of gravity G, supposing the ellipse to revolve on the axis DS; therefore $2cx \times \frac{1}{2}tgc$, or tgc^2x is the generated solid or ellipsoid: but this is also equal to $\frac{2}{3}$ of the circumscribing cylinder (281); that is $tgc^2x = \frac{4}{3}tg^2c$; whence $x = \frac{4g}{3c} = CG$. Now (379, corol. 2), $CG = \frac{\frac{2}{3}CB . AP}{arc\ ABP} = \frac{4g}{3c}$. Therefore the center of gravity of the semicircle ABP, and semiellipse DBS are the same. And in like manner it is found that the center of gravity of the semiellipse ODB is also that of the semicircle described with the radius CD.

392. *If two or more bodies move uniformly in straight lines, their common center of gravity will either be at rest, or move uniformly in a right line.*



1. Let G be the center of gravity of the bodies D and B. Then (372) D : B :: GB : GD. Now if the bodies move from the positions D and B in any right lines whatever, but in opposite directions, and their velocities are as GD and GB, the center of gravity G will remain at rest. For suppose D moves to *d* while B describes B*b*, then the directions being opposite, D*d*, and B*b* are parallel and have the same ratio as GD and GB, and consequently the triangles GD*d*, GB*b* are similar; hence GB : GD :: G*b* : G*d* :: *body* at *d* : *body* at *b*; therefore G is the center of gravity of the bodies when at *d* and *b*.

When D*d* and B*b* coincide with the line passing through D and B, the bodies move directly towards, or from each other.



2. Suppose G the center of gravity of the bodies D and B, as before, and let D be stationary while B moves uniformly from B to *b*; then if GP is parallel to B*b*, the center of gravity G will describe that line GP with an uniform motion in the same time. Again, if B be stationary while D moves uniformly along D*d* in the same time that B described B*b*, G will then describe GO which is parallel to D*d*.

Join *db*, and draw P*g* parallel to D*d* or GO; then the triangles DGP, DB*b*; and also P*bg*, D*bd*, are respectively similar, Hence Dd : GO :: DB : GB :: Db : Pb :: Dd : Pg: now the antecedents Dd, Dd, being the same, the consequents GO, Pg must be equal, and consequently Og is parallel and equal to GP. Moreover, since DB and db are divided proportionally in G and g, the latter point g is the center of gravity of the bodies at d and b.

It therefore follows, that if D and B move uniformly together, and describe Dd, Bb in the same time, their center of gravity G, which is urged in the directions GO, GP, will describe the diagonal Gg of the parallelogram GPgO with the same kind of motion, in that time, whether they move in the same, or different planes; for the points O and P will, in both cases, fall in Bd, and Db, respectively.

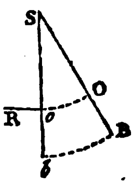
We may now consider D and B as one body at G, moving in the given direction Gg, while a third body describes some other line; and the track of their common center of gravity being determined, as above, a fourth may be added; and so on.

Corol. Hence we conclude that the center of gravity of two or more bodies is not affected by any action of the bodies upon one another. For suppose D and B attract each other, then their center of gravity G is the center of that attraction; the bodies therefore in approaching G must move through spaces proportional to GD and GB, whether they continue in the same line DB, or are urged in the directions Dd, and Db, and consequently G will remain at rest, or describe the line Gg.

It may also be observed, that when a body is projected with a whirling motion, the rotation is made round an axis passing through the center of gravity. So a body if quiescent in free space, may be said to rest on its center of gravity, but an oblique impulse would destroy the equilibrium by turning it on that center.

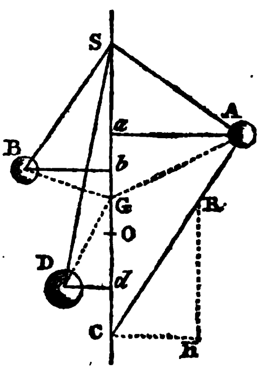
OF THE CENTERS OF PURCUSSION, OSCILLATION, AND GYRATION.

393. THE center of Percussion of a body, or a system of bodies, moving about an axis or point of suspension, is a point which being stopped by an immoveable obstacle, the body or system is quiescent without acting on the axis of motion.



Thus if a rod of wood or metal, SB, vibrating about the end S, strike a fixt obstacle R*o*, and the momenta of *o*S and *ob* are equal (O being in the axis of the rod), then all motion will be destroyed; for neither of the parts *o*S, *ob* would have a tendency to move round the point *o*, which in that case, is the center of percussion, or the point in which all the moving force of the body or rod is collected.

394. *To find the center of percussion* (O) *of a system of bodies* A, B, D, &c. *connected by inflexible lines without gravity, and revolving about the point of suspension* S, *in a plane passing through their centers of gravity.*



Through G the common center of gravity of the bodies, draw SC, upon which let fall the perpendiculars A*a*, B*b*, D*d*, &c. and let AC be perpendicular to SA; also make CR=SA, and draw CH perpendicular and RH parallel to SC. Then AC is the direction of A's motion as it revolves about S; and the system being stopt at O, the body A will urge the point C forward with a force proportional to its velocity into the quantity of matter, that is, as A.SA or A.CR. Now if the force A.CR be resolved into the two forces

A.RH, and A.CH, the latter A.CH will represent the effort of A in a direction perpendicular to SC at the point C; but the triangles CHR, SaA are similar and equal, and therefore $CH = Sa$, consequently $A.Sa$ is the force of A in the direction HC; and (by prop. of the lever) this force drawn into CO, or $A.Sa.CO = Sa.A\,(SC - SO) = Sa.A.SC - Sa.A.SO = A.SA^2 - Sa.A.SO$ is the effort of A to turn the system or bodies about the point O.

In the same manner we get $B.SB^2 - Sb.B.SO$, and $D.SD^2 - Sd.D.SO$, the forces of B and D to turn the mass about the same point O. But when O is quiescent, the forces on contrary sides of that point destroy one another, or their sum is $= o$, that is,

$$A.SA^2 - Sa.A.SO + B.SB^2 - Sb.B.SO + D.SD^2 - Sd.D.SO, \&c. = o;$$

whence $SO = \dfrac{A.SA^2 + B.SB^2 + D.SD^2, \&c.}{Sa.A + Sb.B + Sd.D, \&c.}$, the distance of the center of percussion O from the point of suspension S.

It must be remarked, that when perpendiculars, Aa, Bb, &c. fall on both sides of S, the expressions for those forces which have a tendency to turn the system in a contrary direction, must have contrary signs.

Corol. 1. The common center of gravity of the bodies being G, we have $(A + B + D, \&c.)\,SG = Sa.A + Sb.B + Sd.D, \&c.$ (373. corol. 2.) hence by substitution,

$$SO = \frac{A.SA^2 + B.SB^2 + D.SD^2, \&c}{(A + B + D, \&c.)\,SG}.$$

Corol. 2. But (Geom. art. 83) $SA^2 - Sa^2 = (Aa^2) = GA^2 - Ga^2$; whence

$$SA^2 = GA^2 + Sa^2 - Ga^2 = GA^2 + (Sa + Ga)(Sa - Ga) = GA^2 + SG\,(SG - 2Ga)$$

$$= GA^2 + SG^2 - 2SG.Ga \quad (\text{because } Sa = SG - Ga);$$

that is, $SA^2 = SG^2 + GA^2 - 2SG.Ga$,

In like manner $SB^2 = SG^2 + GB^2 - 2SG.Gb$;

and $SD^2 = SG^2 + GD^2 + 2SG.Gd$, &c.

those values of SA^2, SB^2, &c. being substituted in the preceding fraction which is equal to SO, and its numerator will be

$$= \left\{ \begin{array}{l} A\,(SG^2+GA^2)-(2SG.Ga)A \\ +B\,(SG^2+GB^2)-(2SG.Gb)B \\ +D\,(SG^2+GD^2)+(2SG.Gd)\,D, \text{ \&c.} \end{array} \right.$$

Again, G being the center of gravity of A, B, D, &c. the sum of the products of the bodies by their perpendicular distances from that center on one side, is equal to the sum of the like products on the other (373); that is, $Ga\,.\,A+Gb\,.\,B$, &c. $= Gd\,.\,D$, &c.

Therefore $-(2SG\,.\,Ga)\,A-(2SG\,.\,Gb)\,B+(2SG\,.\,Gd)\,D$, &c. $=0$;

$$\text{hence } SO = \frac{A\,(SG^2+GA^2)+B\,(SG^2+GB^2)+D\,(SG^2+GD^2) \text{ \&c.}}{(A+B+D \text{ \&c.})\,SG}$$

$$= \frac{(A+B+D, \text{ \&c.})\,SG^2+A.GA^2+B.GB^2+D.GD^2, \text{ \&c.}}{(A+B+D, \text{ \&c.})\,SG}:$$

But if we conceive the plane passing through A, B, D, to be the section of any single mass or body, and all the particles of the body reduced to this plane by perpendiculars falling from them upon the plane, then, considering A, B, D, &c. as particles, the sum $A+B+D$, &c. will be the whole mass or body, which put $= b$, and the last expression becomes

$$SO = SG + \frac{A.GA^2+B.GB^2+D.GD^2, \text{ \&c}}{b\,.\,SG}.$$

And $SO-SG\;(=GO) = \dfrac{A.GA^2+B.GB^2+D.GD^2, \text{ \&c.}}{b\,.\,SG}$, the distance of the center of percussion below the center of gravity.

Corol. 3. Hence also, $SG.GO = \dfrac{A.GA^2+B.GB^2+D.GD^2, \text{\&c}}{b.}$; therefore GO is reciprocally as SG, since the bodies A, B, D, &c. and their distances from G are given; consequently if the distance SG is known, GO will also be given.

Corol. 4. If a circle be described about G with the radius GS, the point of suspension (S) may be any where in its circumference, and the distance between the centers of gravity and percussion will continue invariable, the plane of motion remaining as before.

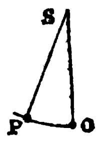
395. *Suppose the body* A (preceding *fig.*) *be made to revolve about* S *by the constant force* f, *acting in a direction perpendicular to* SC, *at a given point* C; *to find the mass, which if placed in* C, *would receive the same angular motion in the same time by the force* f *acting at* C, *as the body receives.*

By considering CSA as a bended lever moveable about S, we have SA : SC :: $f : \frac{f.\mathrm{SC}}{\mathrm{SA}}$ the force at A in equilibrio with the force f at C, or it is the effect of the force f on the point A; the forces f and $\frac{f.\mathrm{SC}}{\mathrm{SA}}$ acting separately at C and A, respectively, would therefore have equal effects on the body A, the former acting at C, and the latter at A.

Let x denote the mass required at C, v and V the velocities of the revolving masses or bodies x and A, respectively, and $\frac{f.\mathrm{SC}}{\mathrm{SA}} = F$. Then by art. 317, (the times being equal), $\frac{v}{V} = \frac{f}{F} \times \frac{W}{w}$, where W and w denote the bodies whose velocities are V and v, that is, $W = \mathrm{A}$, and $w = x$, in the present case. Moreover, when the angular motions of the points C and A are equal, their velocities will be as SC and SA; therefore $\frac{v}{V} = \frac{\mathrm{SC}}{\mathrm{SA}}$, whence by substitution, the equation $\frac{v}{V} = \frac{f}{F} \times \frac{W}{w}$ becomes $\frac{\mathrm{SC}}{\mathrm{SA}} = \frac{\mathrm{SA}}{\mathrm{SC}} \times \frac{\mathrm{A}}{x}$, which gives $x = \frac{\mathrm{SA}^2.\mathrm{A}}{\mathrm{SC}^2}$. And if the force f acts at any other point O instead of C, the mass required will be $\frac{\mathrm{SA}^2.\mathrm{A}}{\mathrm{SO}^2}$.

Corol. 1. In like manner, the masses or bodies $\frac{SB^2 . B}{SO^2}$ and $\frac{SD^2 . D}{SO^2}$ if collected in O, would acquire the same angular motion from any constant force acting at O, as the bodies B and D receive from the same force acting at the same point. Consequently instead of the motion of a system of bodies A, B, D, &c. arising from a force f acting at a given point O, we may consider the motion of the mass $\frac{SA^2.A}{SO^2} + \frac{SB^2.B}{SO^2} + \frac{SD^2.D}{SO^2}$, &c. or $\frac{SA^2.A + SB^2.B + SD^2.D, \&c.}{SO^2}$ when concentrated in O, as an equivalent.

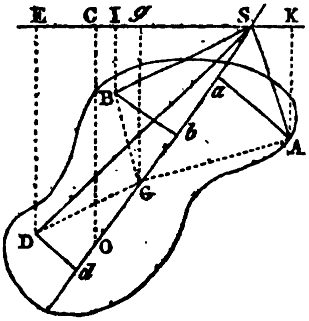
Corol. 2. Let $\frac{SA^2.A + SB^2.B + SD^2.D, \&c.}{SO^2} = m$; then the motive or moving force being f, and m the mass or body moved, the absolute velocity of the point O (or of m, or the whole system A + B + D &c.) will be directly as f, and inversely as m, that is as $\frac{f}{m}$; but the angular velocity is directly as the real or absolute velocity, and reciprocally as the distance SO from the center of motion S*; that is, as $\frac{f}{m}$ divided by SO, or as $\frac{f}{m.SO}$, or $\frac{f.SO}{SA^2.A + SB^2.B + SD^2.D, \&c}$.



* Let m be a body at O moveable about the point of suspension S; then if it be urged through the arc OP by any force f in a certain time t, a greater force would move it through a greater arc in the same time; its velocity therefore will be directly as the moving force f. But if the body be increased, the force f will not be sufficient to urge it through the arc OP in the time t, consequently the velocity, or the space described, diminishes as the body is increased, the velocity therefore will vary as the fraction $\frac{f}{m}$ varies, for a fraction is enlarged by increasing the numerator, but diminished by augmenting the denominator. Again, the angular velocity is measured by the angle OSP or by the arc OP or space described by the body, which space is as the real velocity; but if the radius OS be augmented, the angular velocity is diminished, that is, a less angle is described with the same absolute velocity, in the same time; and therefore the angular velocity is directly as the real velocity, and inversely as the distance OS.

CENTER OF OSCILLATION.

396. THE center of oscillation of a body vibrating by the force of gravity, is that point in which if any quantity of matter be placed, it will perform its vibrations in the same time, and with the same angular velocity as the body itself.



Let G be the center of gravity of the body, DBSAG the plane in which it vibrates, S the point of suspension, O the center of oscillation, ESK an horizontal line, and suppose the matter of the body to be reduced to the plane of vibration by perpendiculars let fall from all its particles A, B, D, &c. upon that plane. Draw A*a*, B*b*, D*d*, &c. perpendicular to S*d* the line passing through the centers of gravity and oscillation, and AK, G*g*, BI, OC, DE, perpendicular to EK.

Since A, B, D, act by the force of gravity in the directions KA, IB, ED, their efforts to move about S, will (by prop. of the lever) be A.SK, B.SI, D.SE; but the effort of A is opposed to that of B and D, and therefore subtractive; whence (corol. 2. preceding art.) the sum B.SI + D.SE — A.SK will be equal to *f*.OS; therefore substituting B.SI + D.SE — A.SK for *f*.OS in the expression denoting the angular velocity (395. corol. 2) we have $\frac{B.SI + D.SE - A.SK}{A.SA^2 + B.SB^2 + D.SD^2}$ the angular motion generated by A, B, D. But if A + B + D, &c. were concentrated in O, the numerator and denominator would become (A + B + D, &c.) SC, and (A + B + D, &c.) SO², respectively; consequently $\frac{(A + B + D)\,SC}{(A + B + D)\,SO^2}$ or $\frac{SC}{SO^2}$ is the angular motion generated by a body at O: now the angular motions are supposed to be equal, therefore $\frac{B.SI+D.SE-A.SK}{A.SA^2+B.SB^2+D.SD^2} = \frac{SC}{SO^2} = \frac{Sg}{SG.SO}$ (from the sim. triang. SCO, SgG), whence $SO = \frac{A.SA^2+B.SB^2+D.SD^2}{B.SI+D.SE-A.SK} \times \frac{Sg}{SG}$. But it follows from art. 373, that $B.SI+D.SE-A.SK = Sg(A+B+D)$ whence, by substitution $SO = \frac{A.SA^2+B.SB^2+D.SD^2}{(A+B+D)\,SG}$; and by the same article, $(A+B+D)\,SG = A.Sa+B.Sb+D.Sd$, therefore $SO = \frac{A.SA^2+B.SB^2+D.SD^2}{A.Sa+B.Sb+D.Sd}$, being the same expression as that for the distance of the center of percussion from the point of suspension. Hence the centers of percussion and oscillation are in the same point. And therefore whatever has been demonstrated in art. 394 respecting the center of percussion, holds equally true for the center of oscillation.

And here it must be observed, that when any of the perpendiculars (Aa, Bb, &c.) fall above the point S, the expressions for the corresponding forces are to be negative.

Corol. 1. If the center of oscillation O be made the point of suspension, S becomes the center of percussion or oscillation, the plane of vibration remaining the same. For let $n = A.GA^2 + B.GB^2 + D.DG^2$; then (394, corol. 2), $\frac{n}{b.SG} = GO$, and $\frac{n}{b.SG} + SG = SO$ the distance of the point of suspension and center of oscillation; therefore if O be the point of suspension, $\frac{n}{b.OG} + OG$ is also the distance of that point from the center of oscillation; but $OG = \frac{n}{b.SG}$, which substituted for OG, and $\frac{n}{b.OG} + OG$ becomes $\frac{n}{b.SG} + SG = SO$, the distance from the point of suspension O to the center of oscillation, as before.

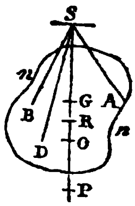
Corol. 2. If p be any particle, as A, B, or D, &c. of the vibrating body, d its distance from the axis of motion S, and b = the body or sum of all the particles $A+B+D$, &c.

then $SO = \frac{A.SA^2 + B.SB^2 + D.SD^2, \&c.}{(A+B+D \&c.)\, SG} = \frac{\text{sum of all the } p.d^2}{\text{body } b \times SG}$ the distance of the center of oscillation from the axis of suspension.

Or if d = the distance of any particle from the center of gravity G, then (394, corol. 2) we have $\frac{\text{sum of all the } p.d^2}{\text{body } b \times SG} = GO$ the distance of the center of oscillation below the center of gravity.

CENTER OF GYRATION.

397. **The** center of gyration of a body, or system of bodies, is that point in which if the whole mass were collected, the same angular velocity would be generated in the same time, by a given force acting at any place, as in the system itself.



Thus suppose the body *nn* to be moved with a certain angular velocity about the axis at S by a force f acting at P, then if all the particles A, B, D, &c. of the body were collected in R the center of gyration, the same force at P would generate, in the same time, an equal angular motion in the mass at R.

To find the point R, we have $\frac{f.SP}{A.SA^2 + B.SB^2 + D.SD^2}$ the angular motion generated in the particles A, B, D, &c. or system, by the force f acting at P (395, corol. 2); but when the system is concentrated in the point R, the expression becomes $\frac{f.SP}{(A+B+D)SR^2}$ for the angular velocity; therefore (by the definition) those expressions are equal, or $\frac{f.SP}{(A+B+D)SR^2} = \frac{f.SP}{A.SA^2 + B.SB^2 + D.SD^2}$, whence $SR = \sqrt{\frac{A.SA^2 + B.SB^2 + D.SD^2}{A+B+D}}$ the distance of the center of gyration R from the axis of suspension at S.

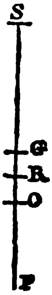
Corol. 1. Since (by the last corol.) $A.SA^2+B.SB^2+D.SD^2 = SO.b.SG$, we get $SR^2 = \frac{SO.b.SG}{b}$, or $SR^2 = SO.SG$, that is, SR is a mean proportional between SO and SG the distances of the centers of oscillation and gravity from the axis of motion.

Corol. 2. If d = the distance from the axis of motion of any particle p of a body b (or $A+B+D$, &c.) then $SR = \sqrt{\frac{\text{sum of all the } p.d^2}{b}}$.

Hence if a body *nn* moves about an axis by the force of gravity, its whole momentum is, or may be considered as in one point O, the center of percussion or oscillation; but when the body is urged by any other extraneous force, that point changes, and is called the center of gyration.

398. *Examples.*

1. To find the center of gyration of a right line or very small cylinder SP, moving about the end S.



Suppose the line or cylinder to be composed of innumerable contiguous particles $p, p, p,$ &c. and $o, m, n,$ &c. their respective distances from S;

Then $po^2+pm^2+pn^2+$ &c......pSP^2
or $p(o^2+m^2+n^2$, &c..........$SP^2)$ = all the pd^2:

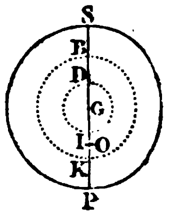
Now (179) the sum of the infinite series of squares o^2+m^2 &c. from o^2 to SP^2, is $\frac{SP^3}{3}$; and since the body $b = SP$, we have $\frac{\text{all the } pd^2}{b} = \frac{pSP^3}{3SP} = \frac{1}{3}SP^2$, (rejecting p as inconsiderable); therefore $SR = \sqrt{\frac{1}{3}SP^2} = SP\sqrt{\frac{1}{3}}$ the distance of the center of gyration R from the axis of suspension S.

2. Let it be required to find the center of percussion or oscillation of the line or small cylinder SP, the axis of motion being at S, as before.

Let G be the middle of SP, or its center of gravity, and O the center of oscillation: then if *p*, *d*, and *b* denote the same as in the preceding example, we have (394. corol. 2) $SO = \frac{pSP^3}{b \times 3SG} = \frac{pSP^3}{SP \times \frac{3}{2}SP} = \frac{2}{3}SP$ the distance of O from the axis of motion.

Corol. Since (356. corol. 3) the length of a simple pendulum vibrating seconds in the latitude of London is 39.13 *inches* $= SO$, therefore $SP = 39.13 + \frac{39.13}{2} = 58.69$ *inches*; which is the length of a small uniform rod that would vibrate by its own weight, once in a second of time; the arcs of vibration being supposed small.

3. To find the center of oscillation of the surface of a circle suspended at the circumference, and vibrating in its own plane.



Let S be the point of suspension, SP a diameter, G the center of the circle or its center of gravity, and O the center of oscillation.

If we suppose the surface of the circle to be composed of the circumferences of innumerable concentric circles DI, BK, &c. and *p* a particle in the circumference at D, or B, &c. then pGD^2 is the product of the particle *p* by the square of its distance from the center of gravity G (396. corol. 2); and (putting $n = 3.1416$), $2nGD$ is the circumference of the circle whose radius is GD; therefore $pGD^2 \times 2nGD$ or $2pnGD^3 =$ all the particles in the circumference drawn into the squares of their distances from G. In like manner $2pnGB^3$ will denote the products in the next circumference, and so on:

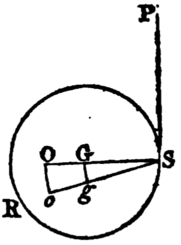
Therefore $2pno^3 + 2pnGD^3 + 2pnGB^3$ &c.........$2pnGS^3$,

or $2pn\,(o^3 + GD^3 + GB^3$ &c,..........$GS^3) =$ sum of all the pd^2 in the surface of the circle SP;

that is, (179) $2pn \times \frac{GS^4}{4}$ or $\frac{pnGS^4}{2} =$ sum of all the pd^2. But *b* (the body) equal area of the circle $= nGS^2$ therefore $\frac{pnGS^4}{2nGS^2.GS} = \frac{GS}{2}$ the distance of the center of oscillation from the center of gravity (396. corol. 2); that is, $SO = \frac{3}{4}$ of the diameter SP.

Corol. Hence if a cylinder be suspended at the circumference of the circular section passing through its center of gravity, and it vibrates in the plane of that section, the center of oscillation will be at the distance of $\frac{3}{4}$ of the cylinder's diameter from the point of suspension. For we may conceive the cylinder to be composed of an infinite number of circular sections or planes.

399. *If one end of a string* PSR, &c. *wrapped round a cylinder, be fastened at* P, *and the cylinder left to descend by its own weight, it will move with a whirling motion; and the space descended, will be to the space described in the same time by a body falling freely, as* 2 *to* 3.



Let RS be the circular section of the cylinder through G its center of gravity, O the center of oscillation, S a momentary point of suspension, and SO parallel to the horizon.

Now if all the matter of the cylinder were concentrated in the point of oscillation O, its angular velocity about the point of suspension S at the beginning of the motion, would be the same as that of the cylinder (396); but the initial velocity of a body at O would be the same as that of a body left to descend freely; hence, if O*o*, G*g* be indefinitely small arcs described by the centers of oscillation and gravity in the same time, their perpendicular velocities (and distances described) will be as the arcs O*o* and G*g*, or as OS and GS; and since the center of oscillation (O) is always in the horizontal line drawn from the point of contact S through the center of gravity G, the velocities of O and G will have the same constant ratio in all stages of the body's descent; but the absolute space descended by the cylinder, is the line described by its center of gravity; therefore, as SG is to SO, so is the perpendicular descent when it turns round its center of gravity, to the space it would describe freely in the same time.

A body descends from rest 16.13 *feet* in the first second of time; therefore SO : SG, or as 3 : 2 :: 16.13 : 10.75 *feet*, the distance which the cylinder would fall in that time by the constant unwinding of the string.

Corol. 1. The tension of the string is $= \frac{1}{3}$ of the weight of the cylinder. For conceive a support at O, then the center of gravity G is prevented from descending by the string at S; consequently, by the nature of the lever, GS : GO (or as $\frac{2}{3} : \frac{1}{3}$) :: *weight* sustained at O : *weight* sustained at S. And this tension is constant; for the point O generates its motion without acting on the point S.

Corol. 2. Hence it appears that when a cylinder rolls down an inclined plane, the space it descends along the plane, is to the space it would describe freely in the same time, were the plane perfectly smooth, as GS to OS. For the forces that generate their motions are both diminished in the ratio of the absolute to the relative gravity upon the plane (343); the spaces described will therefore retain the same ratio, that is, as GS to OS. And the friction on the plane has the same effect on the cylinder's motion, as a string wound round it.

Corol. 3. Since the progressive motion of the cylinder is uniformly accelerated, the rotation about its axis must also be an uniformly accelerated motion.

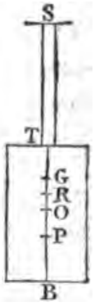
SCHOLIUM. If a simple pendulum and any other body vibrate together in small arcs by their own weight or the force of gravity, and the oscillations are performed in the same time, the length of the pendulum is the distance of the center of oscillation of the body below the point of suspension. A simple pendulum however, is imaginary. But the center of oscillation, or its distance from the axis of suspension may be determined by counting the number of vibrations made in a given time, thus:

Suppose by a good clock or watch a body vibrates 41 times in a minute; then (355), $41^2 : 60^2 :: 39.13 : 83.8$ *inches*, nearly; which is the distance of the center of oscillation of the body below the point of suspension.

In an experiment of this kind, the body should always describe small arcs; and be suspended freely, so that the least force is sufficient to move it.

400. *Suppose* S *to be the axis of suspension of a pendulum* SB, *composed of a block of wood* TB *and a strong bar* ST; *to find the velocity of a bullet, which being discharged against the block at a point* P, *shall cause the pendulum to describe a given arc.*

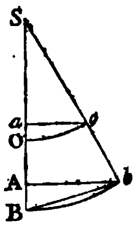
Let G, R, O, be the centers of gravity, gyration, and oscillation of the pendulum; and put SR $= a$, SO $= b$, SP $= c$, the weight of the pendulum $= m$, that of the bullet $= n$, and $x =$ the velocity of the bullet when it strikes the pendulum.



Conceive the whole mass of the pendulum to be collected in the center of gyration R; then (397) the same motion would be generated in the point R by the stroke at P as the pendulum receives; but (395, corol. 1) $\frac{SR^2.m}{SP^2}$ or $\frac{a^2m}{c^2}$ is the mass, which if collected in P, the pendulum would receive the same motion as before, or when all its matter was supposed to be in R. Hence if n and $\frac{a^2m}{c^2}$ are considered as two nonelastic bodies, the former moving with the velocity x, and striking the latter at rest, we shall have (323, cor. 1) $nx \div \left(n + \frac{a^2m}{c^2}\right)$ or $\frac{c^2nx}{c^2n + a^2m}$ the velocity at P with which the pendulum and bullet (as one mass) begin their motion together. But a simple pendulum vibrating in a given arc has the same velocity in the lowest point of that arc as the velocity acquired by a heavy body in its perpendicular descent through the versed sine of the arc (354, corol. 1); and since the velocities acquired by bodies falling freely are as the square roots of the spaces descended (317), if $v =$ the versed sine of the arc described by the center of oscillation O, and $s = 16.13$

feet, it will be $\sqrt{s} : \sqrt{v} :: 2s : 2s\sqrt{\frac{v}{s}} =$ the velocity of O at the lowest point of the arc of vibration; and $b : c :: 2s\sqrt{\frac{v}{s}} : \frac{2c}{b}\sqrt{sv}$ the velocity of the point of impact P when the pendulum begins to move, which therefore must be equal to the former velocity, that is, $\frac{2c}{b}\sqrt{sv} = \frac{c^2nx}{c^2n + a^2m}$; whence $x = \left(1 + \frac{a^2m}{c^2n}\right)\frac{2c}{b}\sqrt{sv}$, the velocity of the bullet when it strikes the pendulum.

This is called the Ballistic Pendulum, contrived by that eminent mathematician Mr. Benj. Robins, for the purpose of determining nearly the initial velocities of shot. We shall give an example in numbers from the pendulum described in his New Principles of Gunnery, Prop. 8.



Let SB be the pendulum in a vertical position, O the center of oscillation, and Bb the arc which B described by the force of the stroke: the chord Bb of this arc was measured by means of a ribon, one end of which was fastened at B.

SB $= 71\frac{1}{8}$ inches, length of the pendulum.
Bb $= 17\frac{1}{4}$ inches, chord of the arc Bb.
SO $= 62\frac{2}{3}$ inches, center of oscillation from the point of suspension.

The chord Bb is a mean proportional between 2SB and the versed sine AB (Geom. art. 216. vol. 1), therefore $\frac{Bb^2}{2SB} = AB$; and the sectors SOo, SBb being similar, we have $SB : SO :: \frac{Bb^2}{2SB} : \frac{SO.Bb^2}{2SB^2} = aO = \frac{62\frac{2}{3} \times (17\frac{1}{4})^2}{2 \times (71\frac{1}{8})^2}$ $= 1.83038$ *inches*, nearly, $= a$O the versed sine of the arc Oo described by the center of oscillation.

Weight of the pendulum .... $56\frac{3}{16}$ *lb.* $= m$
Weight of the bullet .... $\frac{1}{12}$ *lb.* $= n$
Center of oscillation from the point of suspension... $62\frac{2}{3}$ *inches* $= b$
Center of gravity of the pendulum 52 *inches* from the same point,
Whence the distance of the center of gyration $= \sqrt{(52 \times 62\frac{2}{3})}$ $= a$
Point of impact P from the axis of suspension .... 66 *inches* $= c$
16.13 *feet* $=$ 193.56 *in.*..... $= s$
1.83038 *in.*... $= v$

Then, by substitution, $x = \left(1 + \frac{a^2m}{c^2n}\right) \frac{2c}{b} \sqrt{sv} = \left(1 + \frac{52 \times 62\frac{2}{3} \times 56\frac{3}{4}}{66^2 \times \frac{1}{12}}\right) \times \frac{132}{62\frac{2}{3}} \times \sqrt{(193.56 \times 1.83038)} = 20108$ *inches* or 1676 *feet*, the velocity per second, with which the bullet moved when it struck the pendulum. Mr. Robins, by computing with the velocity of the pendulum at the point of impact, instead of the velocity at the center of oscillation, brings out 1641 *feet*: this mistake is noticed by Euler in his comment on Robins's Gunnery.

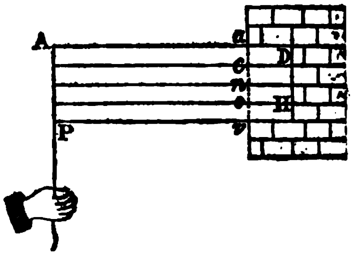
Corol. Since in the same pendulum, and with the same weight of ball, all the quantities in the expression for x are constant, except the versed sine v whose value depends on the length of the chord (Bb), therefore the velocity of the ball is directly as the chord of the arc described by the pendulum.

SCHOLIUM. In the foregoing conclusions it is supposed that the pendulum begins its motion at SB by the stroke of the ball with the same velocity as it acquires in falling back from the position Sb to the perpendicular SB by its own weight: this would be exactly the case did the bullet communicate all its motion to the pendulum at the moment of impact. The ball however, continues to act during the time it is penetrating the wood; and since the motion of the pendulum is circular, and the bullet endeavours to proceed nearly in a right line, its action on the pendulum must produce a shock, or stress on the axis: now both these circumstances may affect the velocity deduced from the rule. A small variation will also result from the augmented weight of the pendulum by the ball, but this is too minute to be of consequence.

OF THE STRENGTH AND STRESS OF TIMBER.

401. THE *lateral strength of squared Timber, is proportional to its breadth drawn into the square of the depth.*

Let PADH represent a vertical section of a beam of timber AH, the end DH being fixed in a wall; and conceive this section to be composed of innumerable parallel fibres *a*, *c*, *n*, *o*, &c.



Now a force P acting perpendicularly at the end AP sufficient to break the beam at *av* will bend it downwards, and the uppermost fibre *a* will be first broken; this done, a less force will bend all the remaining fibres, but the fibre *c* is the next that will break; and then a less force would break the following fibre *n*; and so on: consequently the forces just sufficient to break the fibres in succession will diminish as their number or the depth of the beam is diminished; that is, the strength of the section is as the number of fibres lying upon one another.

Therefore taking DH as the first or greatest force, and calling a fibre *f*, the successive forces will be represented by the infinite arithmetical series DH, DH$-f$, DH$-2f$, DH$-3f$, &c.... to DH$-$DH or *o*: but the number of terms is DH, and therefore the sum of the series will be $(\text{DH}+o)\frac{1}{2}\text{DH}$ or $\frac{1}{2}\text{DH}^2$. Hence if *B* be the breadth or number of perpendicular sections in the beam, its strength or the force necessary to break it, will be as $\frac{1}{2}\text{DH}^2.B$, or as $\text{DH}^2.\text{B}$, because the wholes are proportional to their halves.

Corol. 1. Hence a rectangular beam is stronger with the broadest side vertical than when that side is horizontal, in the proportion of the depth to the breadth. For let D the broadest side be the depth, and B the breadth; then the strength is as D^2B when D is vertical, and as B^2D when it is horizontal; but D and B have the same ratio as D^2B and B^2D.

For example, suppose the depth DH $= 4$ inches, and the breadth $= 1$; and that it can just support a weight at $P = 600$*lb.* then $4^2 \times 1 : 600$*lb.* $:: 1^2 \times 4 : 150$*lb.* the weight it would bear were DH placed horizontal.

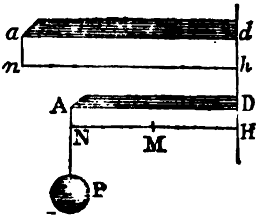
Corol. 2. And the strength of beams of the same depth are as their breadths. For let B and b denote the breadths, and D the common depth, then D^2B and D^2b will represent the strengths, which expressions are as B and b.

Corol. 3. Hence also, the lateral strengths of beams whose sections are similar, will be as the cubes of their breadths or depths: thus if D^2B, and d^2b denote the strengths, then the sides being proportional, we have $D : B :: d : \frac{Bd}{D} = b$, which substituted for b, and d^2b becomes $\frac{Bd^3}{D}$; now D^2B and $\frac{Bd^3}{D}$ multiplied by D, give D^3B and Bd^3, which are as D^3 and d^3.

Corol. 4. Also, since D^2B and d^2b are the areas of the sections multiplied by the depths; therefore the strengths of beams having similar sections, will be as their areas multiplied by the depths: Or as the cubes of the depths, when those depths are homologous.

Thus, if a cylinder AH whose diameter DH is 4 inches, can just sustain a force at $P = 800$*l.* then a cylinder of the same material, 1 inch in diameter, and of the same length, will bear only $12\frac{1}{2}$*lb.* for $4^3 : 1^3 :: 800 : 12\frac{1}{2}$.

402. *If the beam* AH *of a given length, and depth, when fixed horizontally at the end* DH, *can just support a given weight P at the other end; to find the dimensions of a similar beam* (*ah*) *of the same material, that will break by its own weight, or only just sustain itself.*



Suppose the beam to be of uniform thickness, and let $W =$ its weight. Then we may consider the end DH as the fulcrum of a lever NH void of gravity, supporting a weight at N equal to P, and another weight $= W$ at M the middle of NH directly under the center of gravity of the beam. And by the nature of the lever, the effort of the weight W to bend or break the lever at DH, will be as $MH.W$, or $\frac{1}{2}NH.W$, and that of the weight P as $NH.P$, therefore $\frac{1}{2}NH.W + NH.P$, or $(\frac{1}{2}W + P)\,NH$ is the whole effort, or stress on the fulcrum DH.

But since the beams are similar, their weights will be as the cubes of the lengths, or depths; hence $DH^3 : dh^3 :: W : \frac{dh^3.W}{DH^3} =$ the weight of the beam ah; and $DH : NH :: dh : \frac{NH.dh}{DH}$ its length; therefore $\frac{dh^3.W}{2DH^3} \times \frac{NH.dh}{DH}$, or $\frac{dh^4.W.NH}{2DH^4}$ is the stress of the beam ah on the end or fulcrum $dh : dh$.

Now the strengths of the beams must be as the stresses, but the strengths are as DH^3 and dh^3; therefore $DH^3 : dh^3 :: (\frac{1}{2}W + P)NH : \frac{dh^4.W.NH}{2DH^4}$, whence $W : W + 2P : DH : dh$ the depth of the beam; and $DH : NH :: dh : nh$ its length.

Let the ends of AH be squares, the side $DH = 1$ *inch*, length $NH = 1$ *foot*, its weight $= .3$*lb.* and the weight $P = 100$*lb.*

then $W : W + 2P :: DH : dh$,

or $.3 : .3 + 200 :: \frac{1}{12} : 55.14$ *feet*, nearly, $= dh$; and $55.14 \times 12 = 667.67$ *feet*, nearly, the length nh.

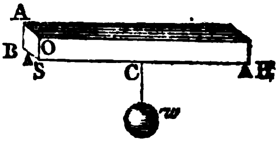
Corol. Hence, if it be required to find the length of a spar having the same depth and breadth as AH, that would break by its own weight, let $l =$ the length in feet, then $.3l =$ the weight in *lbs.* and $\frac{1}{2}l \times .3l$ is the effort of its own weight; therefore $\frac{1}{2}l \times .3l = (\frac{1}{2}W + P)\,NH$ the effort of AH (together with the weight P) on the end or fulcrum DH,

That is $.15l^2 = (\frac{1}{2} \times .3 + 100) \times 1 = 100.15$, and $l = \sqrt{\frac{100.15}{.15}} = 25.8$ &c. *feet*, the length required.

SCHOLIUM. From these computations it appears that in the construction of works, &c. it is possible to take a beam of such dimensions that the stress by its own weight may exceed its strength. Machines may also be made too large to be useful, for the less are stronger in proportion to their bulk than the greater when the dimensions of both are similar. Thus we find that small animals are stronger and more active in proportion to their weight or size than large ones.

403. *It is found by experiment that a spar of oak* (AH) *an inch square, and* 1 *foot in length* (SH), *when supported horizontally at the ends, will bear about* 670lb. (w) *suspended at the middle* (C) *before it breaks; hence it is required to find what weight a piece of the same oak will bear, which is* 10 *feet long,* $\frac{1}{2}$ *a foot deep, and* $\frac{1}{3}$ *of a foot broad, the weight being also suspended at the middle?*

Let the depth OS $= d$, depth $\frac{1}{2}$ foot $= D$,
breadth OA $= b$, breadth $\frac{1}{3}$ foot $= B$,
length SH $= l$. length 10 feet $= L$,
required weight $= W$.



Then d^2b will denote the lateral strength of the spar AH, and D^2B that of the other piece. And since each prop sustains a weight $= \frac{1}{2}w$ acting at C, if C were the fulcrum, and a force $= \frac{1}{2}w$ acted vertically at each end A and H, their efforts to break the spar would be the same as that of the weight w when the spar is supported on the props; but (by prop. of the lever) the effort of $\frac{1}{2}w$ is $\frac{1}{2}w \times \frac{1}{2}l$, and $\frac{1}{2}W \times \frac{1}{2}L$ that of $\frac{1}{2}W$; and since the two pieces just support the weights, their strengths must be as the efforts or the greatest forces they resist,

that is, $d^2b : D^2B :: \frac{1}{2}w \times \frac{1}{2}l : \frac{1}{2}W \times \frac{1}{2}L$,

whence $d^2bWL = D^2Bwl$, and $W = \frac{D^2Bwl}{d^2bL} = \frac{\frac{1}{4} \times \frac{1}{3} \times 670 \times 1}{\frac{1}{144} \times \frac{1}{12} \times 10} = 9648$lb the *answer*.

*403. *If the spar* AH (preceding art.) *break with* 660lb. *suspended at* C; *then what will be the length of another piece of the same wood,* $\frac{1}{2}$ *a foot square, that will support* 17820lb, *at its middle?*

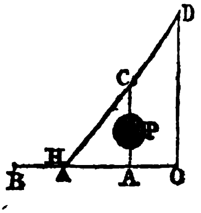
Here $d=\frac{1}{12}$, $b=\frac{1}{12}$, $l=1$, $w=660$, $D=\frac{1}{2}$, $B=\frac{1}{2}$, $W=17820$. And from the equation $d^2bWL=D^2Bwl$, we get

$$L=\frac{D^2Bwl}{d^2bW}=\frac{\frac{1}{8}\times 660\times 1}{\frac{1}{1728}\times 17820}=8 \textit{ feet},$$ the length required.

404. ***A deal spar* 1 *foot long, and an inch square, when supported horizontally at the ends, will bear about* 500lb. *suspended at the middle; then what weight will a plank* 3 *inches deep, and* 10 *inches wide, sustain in the same position, if it rest on two props* 9 *feet asunder.***

In this example $d=\frac{1}{12}$, $b=\frac{1}{12}$, $l=1$, $w=500$, $D=\frac{3}{12}$, $B=\frac{10}{12}$, $L=9$, which substituted in the equation $W=\frac{D^2Bwl}{d^2bL}$, gives $W=2500$*lb.* the required weight. And the same equation or theorem will serve for comparing the strengths of prisms, or bars of metal one with another.

405. ***If the beam* HD *in a position oblique to the horizon* HO *be loaded with a weight* P *at the center* C, *and* CA, DO *perpendicular to* HO; *then the stress at* H *(or* D) *is as* AH × *P*.**

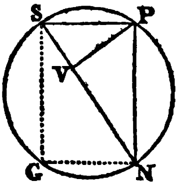


For suppose the horizontal line HB to be connected with HD at H, then BHC may be considered as a bended lever where the force *P* acts perpendicular to the horizon BO, and therefore (359, corol. 2) AH × *P* is the effort of *P* to turn the lever about the fulcrum H.

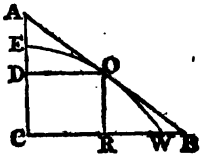
Corol. Hence if HO were another beam supported at the ends H and O, and of the same material and thickness as HD, the two beams would require just the same weight at the centers C and A to break them.

When the beams are large, it may be necessary to make an allowance by including their weight in the pressure, &c.

406. ***To cut the strongest scantling or rectangular beam of timber* SPNG *from a cylindric one.***



Let the circle whose diameter is SN be a section of the cylinder. Put $d = $ SN, and y $=$ SP the breadth of the beam; then $d^2 - y^2$ $=$ PN^2 the square of the depth: and since the strength is as the breadth into the square of the depth, it will be denoted by $(d^2 - y^2)y$ or $d^2y - y^3$, which is to be the greatest possible, or a *maximum*.



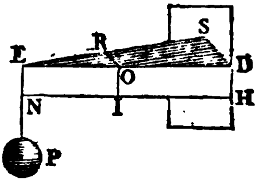
Suppose ED an absciss, DO the corresponding ordinate of the parabola EOW, OA a tangent at O, OB = OA, and CB parallel to DO. Then because AB is bisected in O, the rectangle DR is a maximum or the greatest that can be inscribed in the parabola, or in the triangle CAB, (245, corol. 2); and since ED = EA (297, corol. 2), and DC = DA, therefore ED = $\frac{1}{3}$EC.

If the absciss ED $= x$, ordinate DO $= y$, $p =$ the parameter, and $a =$ EC; then $a - x =$ DC, and $(a - x)y =$ the rectangle DR; but $px = y^2$ (by prop. of the parabola) and $x = \frac{y^2}{p}$ which substituted for x, and $(a - x)y$ will be $\left(a - \frac{y^2}{p}\right)y$: let the parameter $p = 1$, then $\left(a - \frac{y^2}{p}\right)y$ becomes $ay - y^3 =$ the rectangle DR; but when $p = 1$, then $x = y^2$, therefore $y^2 = \frac{1}{3}a$, that is, when $ay - y^3$ is a maximum, $y^2 = \frac{1}{3}a$, and therefore the expression $d^2y - y^3$ is a maximum when $y^2 = \frac{1}{3}d^2$, or SP$^2 = \frac{1}{3}$SN^2, and consequently PN$^2 = \frac{2}{3}$SN^2; that is, SP and PN are in the same ratio as the side of a square and its diagonal. Hence this construction; Make SV $= \frac{1}{3}$SN, and erect the perpendicular VP; then PS is the breadth of the rectangular end. For VP^2 $=$ SV $\times$ VN $= \frac{1}{3}$SN $\times$ $\frac{2}{3}$SN $= \frac{2}{9}$SN^2, and PS$^2 =$ SV$^2 +$ VP^2 $= \frac{1}{9}$SN$^2 + \frac{2}{9}$SN$^2 = \frac{1}{3}$SN^2.

Corol. Because SV.VN $=$ VP^2, therefore SV.VN^2 + VN^2 $=$ PN^2. Also by sim. triangles, SN : SP :: SP : SV, whence

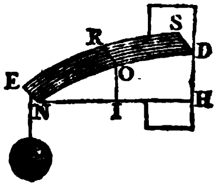
$\sqrt{(SN.SV)} = SP$; and the maximum $SP.PN^2 = \ldots\ldots\ldots$ $\sqrt{(SN.SV)}\,(SV.VN+VN^2) = \sqrt{(SN.SV.VN^2)}\,(SV+VN)$; but $SV+VN$ is given, therefore $\sqrt{(SN.SV.VN^2)}$, or its square $SN.SV.VN^2$ is also a maximum; consequently (because SN is given) $SV.VN^2$ is a maximum: hence when a given line (SN) is divided so, that the solid under one part and the square of the other, is the greatest possible, the least part (SV) will be half the other (VN).

407. *Suppose a weight P is supported at one end of a beam* EH *of a given depth* EN *or* DH, *the other end* SDH *being fixed in a wall; to find the figure of the beam when its strength at any vertical section* (IOR) *is equal to the stress at that section: Or that it shall be as liable to break at any one place as at another.*



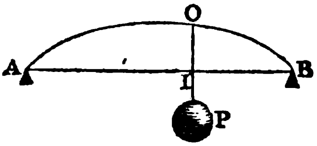
Let the section IOR be parallel to the end SDH; then (401, corol. 2) the strengths at IOR and HDS, are as OR and DS; and the stress at IOR and HDS, as EO × P and ED × P, or as EO and ED; but the strength is supposed to be equal to the stress; therefore OR and DS have the same ratio as EO and ED, and consequently ESD is a triangle; therefore EH is a prism; the upper side ESD (and also the lower) being parallel to the horizon.

408. *To determine the figure of the beam when the breadth* EN *or* SD *is given.*



If the sections IOR, SDH are parallel, as in the last article, the strengths or stresses at these sections will be as $IO^2 \times OR$ and $HD^2 \times DS$, or as IO^2 and HD^2, because $OR = DS$; but the stresses are as NI and NH, that is, the squares of IO and HD are as NI and NH, and therefore NOD is the curve of a common parabola whose axis is NH, and vertex N.

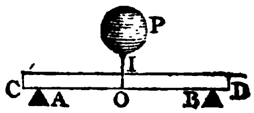
409. *Let* AOBI *be a vertical section or one side of a beam of timber of a given breadth, supported horizontally on the props at* A *and* B; *to find the figure of the beam so that its strength shall be as the stress when a given weight* (P) *is suspended any where between the ends* A, B.



Since the weight P is supported by both props, we have (384) $AB : P :: IB : \frac{IB.P}{AB}$ = that part sustained by A, and $\frac{IA.P}{AB}$ the other part sustained by B; and therefore $IA \times \frac{IB.P}{AB}$ is the stress at IO arising from the weight $\frac{IB.P}{AB}$, and $IB \times \frac{IA.P}{AB}$ the stress from the other weight, and the sum of both, or $IA \times \frac{IB.P}{AB} + IB \times \frac{IA.P}{AB} = IA \times IB \times \frac{2P}{AB}$ is the whole stress at IO; but $\frac{2P}{AB}$ is given, therefore the stress every where is as the rectangle $IA \times IB$.

Now the beam being of an equal breadth, its strength will be as the square of the depth, or as OI^2; but the strength must be as the stress, that is, OI^2 is as $IA \times IB$, or the rectangle of the two abscisses, is as the square of the corresponding ordinate. Therefore the section OIB is a semiellipse, (267).

410. *If the beam* CD *when resting loose on the props* A *and* B *near the ends, can just support a weight P at the middle, it will bear double that weight when the ends* C, D *are fixed down so as to be immoveable.*



When the ends are loose on the props, each bears half the weight, and the beam will break in the middle I where the stress is greatest. Hence, if the supports A and B were removed, and the beam rested on a prop at O, it follows that half the weight P suspended at each of the points A and B would also break the beam at the middle O, that is, the pressure on the fulcrum O when the beam broke, would be equal to the weight P. Now if the ends C and D are fastened down, and the weight P doubled, each of the props A and B will support half the weight, or a weight equal to P, therefore double the weight P would just break the beam at A and B when the ends are fixed.

Or thus. When the ends are fixed, we may conceive forces acting at C and D sufficient to produce an equilibrium on the fulcra A and B, so that $CA \times$ *force at* $C = AO \times \frac{1}{2}P$, and $DB \times$ *force at* $D = BO \times \frac{1}{2}P$, and hence it appears than an additional weight equal to P will be necessary to produce a like effect.

Remark. Pieces of wood of equal dimensions, cut from the same beam or plank, are found to differ in strength, and therefore computations from theory seldom agree with experiment.

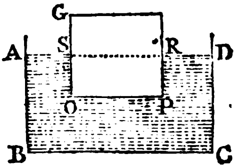
By experiments made on deal spars an inch square, resting loose on two props 1 foot asunder, it was found that they bore from 460 to 690*lb.* at the middle before they broke. Oak spars from 660 to 970*lb.* And cast iron bars from 730 to 990*lb.* the bars being an inch square, and the supports 3 feet apart.

HYDROSTATICS.

411. IF *a body of uniform density of the same specific gravity as water, or whose weight is equal to the weight of the same bulk of water, be immersed in that fluid, it will rest in any position.*

For when the specific gravities of the body and fluid are equal, the latter is pressed by the former just as much as it is by the like bulk of water when it fills the space occupied by the body. The body therefore in any position can have no more tendency to rise or sink than the water itself.

412. *A body heavier than the fluid will sink to the bottom. But if it be lighter it will float with only a part immersed.*

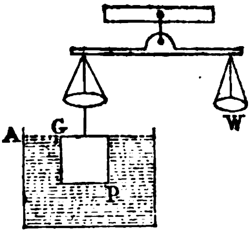


Let AD be the surface of the water in the vessel AC. Then if the body GP be heavier than the fluid, its pressure on the water underneath is greater than that of an equal bulk of water, or the resistance in the fluid is less than the force by which the body endeavours to descend; and since the parts of fluids are easily moved among themselves, the body will sink by its superior gravity.

But when the specific gravity of the body is less than that of the water, the body can only sink till the weight of the fluid displaced is equal to the weight of the body, or till the pressure and resistance are equal. Thus if GP be a cube of wood weighing 500 *ounces* avoirdupois, and the side OP or OG a foot, it will sink just 6 inches in spring water, which weighs 1000 *ounces* per cubic foot. Hence it appears that fluids press upwards as well as downwards, for the action of the water in a vertical direction under the body GP is equal to the compressing force or pressure of the body downwards.

If the weight of the cube were 1000 *ounces*, it would sink ’till its upper surface became level with the surface of the water, and the water would exert a force against the bottom OP equal to the weight of the cube. The pressure of fluids vertically is therefore the same as in the direction of gravity at the same depth; hence a body lighter than the fluid, when immersed, will rise till it floats at the surface, for the force of the fluid against the body underneath is greater than its weight together with that of the column of water directly over it.

413. *Bodies suspended in a fluid lose as much weight as the weight of the fluid displaced.*



Thus the cube GP whose side is a foot, and its weight = 1000 *ounces*, when put in water, loses all its weight, that is, it will require no force to keep it suspended in the fluid, because it is just as heavy as the same bulk of water. But if the cube were of brick (the specific gravity being double that of water) its weight would be 2000 *ounces*, and a counterpoise = 1000 *ounces* in the scale W would be necessary to keep it from sinking, because in that case the water sustains half the weight of the cube. A body of lead of the same dimensions is 11325 *ounces*, therefore its weight would be 11325 — 1000 or 10325 *ounces* when weighed in water. But a cubic foot of cork which weighs only 240 *ounces* would require a weight on the upper side, or a force equivalent to 1000 — 240 or 760 *ounces* to sink it even with the surface of the water.

Corol. Hence to determine the specific gravity of a body heavier than water; first let it be weighed in air, after the usual method, then weigh it in water (which may be done by suspending it from the balance or scales), and the difference of the results will be the weight of water equal in bulk to the body; and that difference, and the weight of the body in air, will be the specific gravities of water and the body: for the specific gravities of bodies are denoted by their weights when the magnitudes are equal.

Thus suppose a mass of lead to weigh $226\frac{1}{2}$ and $206\frac{1}{2}$ *ounces*, respectively, in air, and in water, then $226\frac{1}{2}-206\frac{1}{2}=20$ *ounces* is the weight of a volume of water equal in magnitude to the lead; hence the specific gravity of water to that of lead is as 20 to $226\frac{1}{2}$, or as 1000 to 11325.

Hence, if A = the absolute weight of a body,
a = its weight in water,
s = its specific gravity;
w = the specific gravity of water.

Then $A-a$ is the weight of a volume of water equal in magnitude to the body:

And $A-a : A :: w : \frac{Aw}{A-a} = s$, the specific gravity of the body A.

The value of w, which denotes the specific gravity of water, is arbitrary. In some tables it is supposed to be 1. But it is much more convenient to make it $= 1000$, because a cubic foot of pure water weighs 1000 *ounces* avoirdupois.

414. *To find the specific gravity of a body (B) lighter than water.*

Let a heavy body whose weight is A be attached to it, so that both will sink together;

and put $B =$ the weight of the body B,
$r =$ its specific gravity,
$c =$ the weight of the compound in water,
$a =$ the weight of A in water,
$w =$ the specific gravity of water:

Then, proceeding as in the last corollary, we have
$c-a$, the weight of the lighter body B in water, which, in this case, is negative because a is greater than c.
$B-(c-a)$ or $B+a-c$, the weight of a volume of water equal in magnitude to the body B:

And $B+a-c : B :: w : \frac{Bw}{B+a-c} = r$, the specific gravity of B.

Example. Suppose 755 ounces of lead is attached to a block of deal weighing 275 ounces, and that the weight of the whole together in water is $463\frac{1}{3}$ ounces. What is the specific gravity of the deal?

$11325 : 10325 :: 755 : 688\frac{1}{3}$ ounces, the weight of the lead in water:

$B = 275$ $\qquad a = 688\frac{1}{3}$
$c = 463\frac{1}{3}$ $\qquad w = 1000.$

And $\frac{Bw}{B+a-c} = \frac{275 \times 1000}{275 \times 688\frac{1}{3} - 463\frac{1}{3}} = 550$, the specific gravity required.

This result is *ounces:* and it is equal in bulk to 1000 ounces of water with which it is compared. The wood therefore weighs 550 ounces per cubic foot.

415. *The weights of bodies are proportional to the products of their masses and specific gravities.*

Thus if a mass of lead be l cubic inches; then since a cubic foot or 1728 cubic inches weigh 11325 ounces,

we have, $1728 : 11325 :: l : \frac{11325l}{1728}$, weight of the mass.

Or if $d =$ the cubic inches in a piece of deal,

then $1728 : 550$ *ounces* $:: d : \frac{550d}{1728}$ its weight in ounces;

But 11325 and 550 denote the specific gravities of lead and deal: and $11325l$ and $550d$ are in the same ratio as $\frac{11325l}{1728}$ and $\frac{550d}{1728}$.

416. *To determine the quantities in a mass compounded of two ingredients when its weight, and the specific gravities are given.*

Let A and B denote the magnitudes of the two ingredients,
a and b their specific gravities, respectively;
C the weight of the compound, or the weight of $A+B$,
c its specific gravity:

Then by the last article, aA, bB, and $cA+cB$, will denote, or be proportional to the weights of A, B, and $A+B$, respectively;

$$\text{consequently } aA+bB=cA+cB$$
$$\text{or } aA-cA=cB-bB,$$

whence $A : B :: c-b : a-c$, that is, the magnitudes A and B are as $c-b$ and $a-c$:

And therefore the weights of A and B will be as $a(c-b)$ and $b(a-c)$, or as $ac-ab$ and $ab-ac$:

Consequently we have to divide the weight C into two parts having the proportion of $ac-ab$ to $ab-ac$:

Therefore

$$ac-ab+ab-bc : C :: ac-ab :$$

$$\text{or}\quad ac-bc : C :: ac-ab : \frac{(c-b)\,aC}{(a-b)\,c}, \text{ the weight of } A,$$

$$ac-bc : C :: ab-bc : \frac{(a-c)\,bC}{(a-b)\,c}, \text{ the weight of } B.$$

Example. Suppose a composition of Copper and Tin to be 42*lb.* and its specific gravity 8000; what is the quantity of each metal, the specific gravity of Copper being 9000, and that of Tin 7320?

$a=9000$ $\qquad$ $c=8000$

$b=7320$ $\qquad$ $C=\ 42.$

And $\frac{(c-b)\,aC}{(a-b)\,c}=19\frac{1}{8}$*lb.* the weight of Copper. Consequently $42-19\frac{1}{8}$ $=22\frac{7}{8}$*lb.* the weight of Tin.

Corol. Hence we can find the specific gravity of a mass compounded of two ingredients when their weights and specific gravities are given. For suppose A and B to denote the weights of A and B, respectively: then $\frac{(c-b)\,aC}{(a-b)\,c}=A$, whence $\frac{(A+B)\,ab}{bA+aB}$ $=c$, the specific gravity required.

And in the same manner a, or b, may be found when the values of the other letters are given.

417. *To find the specific gravity of a fluid.* Let a body whose specific gravity is known, be weighed both in, and out of the fluid, then the specific gravity may be found from the expression $\frac{Aa}{A-a}=s$ (Art. 413, corol.) or $w=\frac{(A-a)\,s}{A}$, where w denotes the specific gravity of the fluid:

Thus suppose 28*lb.* 5 *ounces* of lead when weighed in water, is 25*lb.* 13 *ounces*; then $A=28\frac{5}{13}$, $a=25\frac{13}{16}$, and $s=11325$ the specific gravity of lead;

and $w=\frac{(A-a)\,s}{A}=1000$, the specific gravity of water.

Corol. Hence to compare the specific gravities of two fluids, weigh the same body in both, and we shall have,

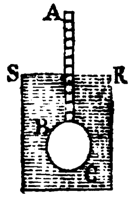
As the loss of weight in one fluid, is to the loss in the other,

So is the specific gravity of the former fluid, to that of the other:

Thus if $31\frac{1}{4}$ *ounces* be the loss in water, and $27\frac{1}{16}$ *ounces* the loss is rectified spirit of wine, the specific gravities of the fluids are as $31\frac{1}{4}$ and $27\frac{1}{16}$, or as 1000 to 866 which therefore is the specific gravity of the spirit, that of water being 1000.

Platina and Gold are the only known substances that will sink in Quicksilver or Mercury: but its specific gravity may be determined by putting it in a small open glass vessel suspended from the scale, and weighing it in water: the vessel however, must be first balanced in water,

418. The instrument used in these experiments for weighing, is an *Hydrostatical Balance,* which however, differs but little in the construction from a pair of scales. But for comparing the specific gravities of fluids that are nearly of the same density, another instrument has been contrived, called the *Hydrometer*. The common sort consist of a graduated cylindric stem AB fixed to a hollow globe of copper BC; its weight being adjusted so, that it may swim in the fluid with part of the stem above the surface SR. When this is put in proof spirits (for example) it will sink to a particular division on the stem; but if the spirit be under proof or weaker than proof spirit, (as common brandies, &c.) it will not sink to that division; on the contrary, should the fluid be rectified spirit of wine, the hydrometer will descend to a division on the stem much above the proof point,



Scholium. The specific gravity of bodies however, must necessarily vary with their temperature, for most bodies expand by heat, but are contracted again in cooling. And in computing the specific gravity of a compound, an error may arise in consequence of some uncertainty in its bulk. Thus a pint of spirits of wine and another of water when mixed together will be less than a quart. To account for this diminution, or *penetration of dimensions*, it has been supposed that the constituent particles of the fluids are globular, but of a size much less in one fluid than in the other, because in that case, a considerable number of the less particles might fall into the spaces between the greater particles in the operation of mixing. Mr. Ramsden, in 1702, published an account of a new instrument called the *Balance Hydrometer*, by which he could determine the exact quantity of spirit and of water in any compound of the two, its specific gravity, the diminution in the volume, &c. &c. at any temperature.

419. *A Table of Specific Gravities.*

Platina	22000	Nitre	1900
Fine Gold	19400	Ivory	1825
Standard Gold	17724	Sulphur	1810
Pure Mercury	14000	Chalk	1790
Common Mercury	13600	Dry Lignum vitæ	1327
Lead	11325	Pit Coal	1250
Fine Silver	11091	Dry Mahogany	1063
Standard Silver	10535	Dry Boxwood	1030
Copper	9000	Sea Water	1030
Copper halfpence	8915	Spring Water	1000
Gun metal	8784	Distilled Water	993
Cast Brass	8000	Gunpowder close shaken	937
Steel	7850	Proof Spirits, in the temperature of 55°	927
Wrought Iron	7645		
Cast Iron	7425	Dry Oak	925
Tin	7320	Ice	900
Diamond	3517	Rectified Spirits of Wine	866
Marble	2700	Dry Ash	800
Common Green Glass	2600	Beech	700
Flint	2570	Elm	600
Common Stone	2520	Fir or Deal	550
Clay	2160	Cork	240
Brick, from 2000 to	2400	Air in a mean state	$1\frac{1}{5}$
Common Earth	1984		or $1\frac{2}{9}$

The numbers in the foregoing table denote the specific gravities of the several bodies, and also the weight of a cubic foot of each in *ounces* avoirdupoise. Thus a cubic foot of common water weighs 1000 *ounces* or $62\frac{1}{2}$*lb.* A cubic foot of cast iron 7425 *ounces*, &c. &c. Hence we have a ready method of finding the magnitude of a body from its weight, or the weight from its magnitude, as in the following examples.

1. Suppose an irregular piece of *Oak timber* to weigh $14\frac{1}{2}$ hundred weight; how many cubic feet does it contain?

$14\frac{1}{2} \times 112 = 1624$*lb.* $= 25984$ *ounces*, the weight, which divided by 925 (the ounces per cubic foot) gives 28 $\frac{84}{925}$ cubic feet, the *answer*.

2. What is the weight of 600 square feet of *sheet Lead* $\frac{1}{10}$ of an inch thick?

$\frac{1}{10} \times \frac{1}{12} = \frac{1}{120}$ of a foot, the thickness:
and $\frac{1}{120} \times 600 = 5$ cubic feet, the content:

Then, as 1 *foot* : 11325 *ounces* :: 5 : 56625 *ounces* the weight, which divided by 16 gives $3539\frac{1}{16}$*lb.* the *answer*.

3. What is the diameter of a cast *Iron Shot* whose weight is 9*lb.*?

9*lb.* $\times 16 = 144$ *ounces:*

Then, 7425 : 1728 *in.* :: 144 : 33.5127 *inches* nearly, the cubic contents:

And .5236 being the contents of a sphere whose diameter is 1, we have .5236 : 1^3 :: 33.5137 : 64 *inches*, very nearly, the cube of the diameter; Therefore the diameter of a 9*lb.* iron shot may be taken at 4 inches without sensible error.

4. To find the weight of a *Lead Shot* of a given diameter.

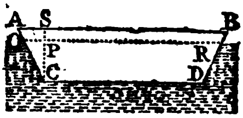
Put $d =$ the diameter in inches; then $.5236d^3$ is the cubic contents in inches:

And 1728 *in.* 11325 *oz.* :: $.5236d^3 : \frac{11325 \times .5236d^3}{1728}$, or $\frac{5930d^3}{1728}$ nearly, is the weight in ounces, of the ball whose diameter is d; hence if $d = 12$ inches the weight will be 5930 *ounces*.

If the diameter $= 3.6$ or $3\frac{3}{5}$ *inches*, the weight is 10*lb.* or 160 *ounces*, nearly; for $12^3 : 5930 : (3\frac{3}{5})^3 : 160$; which proportion is more exact than that given in the Arith. *art.* 188, examp. 7.

Or the cube of the diameter in *inches*, multiplied by 3.432 gives the weight in *ounces*. And if the weight in *ounces* be multiplied by the decimal .2914, the cube root of the product will be the diameter in *inches*.

5. Let ABDC be the profil of a *Pontoon* floating in water, the water mark being OR. Put $n =$ CD the external length at bottom, $r =$ the difference of CD and the length at top AB, $d =$ SC the depth, $s =$ PC the depth of the part immersed, and $b =$ the breadth.



By similar triangles, $d : \frac{1}{2}r$ (or AS) $:: s : \frac{sr}{2d} =$ OP, whence $n + \frac{sr}{d} =$ OR, and $\left(n + \frac{sr}{2d}\right) s =$ the area of the trapezoid OCDR, therefore $\left(n + \frac{sr}{2d}\right) sb$ is the cubic contents of the immersed part. Suppose the dimensions are in inches, and let $f = 1728$, $l = 62\frac{1}{2}$*lb.* avoirdupoise the weight of 1728 cubic inches of water, and $w =$ the weight (in pounds) of the water displaced, or the weight of the pontoon together with the weight it bears; then $\left(n + \frac{sr}{2d}\right) \frac{sbl}{f}$, or $\frac{nbl}{f} s + \frac{rbl}{2df} s^2 = w$.

If the weight w be given, and the depth PC required, the equation gives $s = \sqrt{\left(\frac{2dfw}{rbl} + \frac{d^2n^2}{r^2}\right)} - \frac{dn}{r} =$ PC.

Let the outward dimensions be

AB $= 21\frac{1}{2}$ feet,
CD $= 17\frac{1}{6}$............ $= 206$ *in.* $= n$,
SC $= 2\frac{1}{4}$ .... $= 27$ *in.* $= d$, depth,
$r = 4\frac{1}{3}$ .... $= 52$ *in.*
breadth $= 4\frac{3}{4}$ .... $= 57$ *in.* $= b$:

And suppose the water mark OR to be 9 inches from the top, or PC $= 18$ *in.* $= s$;

Then $\frac{nbl}{f} s + \frac{rbl}{2df} s^2 = 8288$*lb.* $= w$, the weight (including the pontoon) that sinks it 18 inches.

Again, if the weight, including the pontoon $= 6000$*lb.* $= w$; what is the depth in the water.

$$s = \sqrt{\left(\frac{2dfw}{rbl} + \frac{d^2n^2}{r^2}\right)} - \frac{dn}{r} = 13.3 \text{ inches} = \text{PC, the depth required.}$$

Hence, for pontoons of the above dimensions, the expressions will become

$$424.7s + 1.985s^2 = w.$$

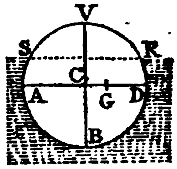
And $\sqrt{(.50371w + 11441)} - 107 = s =$ PC.

That is, multiply the depth sunk (in *inches*) by 424.7, and its square by 1.985, and the sum of the products is the weight of the loaded pontoon in *pounds*.

And to find the depth when the weight (including the pontoon) is given, Multiply the weight in *lbs.* by the decimal .50371, and add the product to 11441, then 107 subtracted from the square root of the sum, gives the depth PC in *inches.*

A pontoon of the above dimensions weighs about 900*lb.*

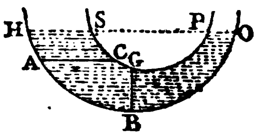
420. *If a body is at rest when floating in a fluid, its center of gravity and the center of gravity of the fluid displaced, are in the same vertical line.*



Suppose the body to be a globe whose center is C, and let its density be unequal so that G (instead of the center C) is its center of gravity; then since the centre of gravity of the fluid displaced or segment SRB, is in the vertical diameter VB, the center of pressure of the fluid upwards is in that vertical line, the center C may therefore be considered as the point upon which the body is suspended, consequently it must move round that point till the diameter AD becomes vertical. (372, corol. 1 and 2). And the same method of reasoning will apply to bodies of any form.

Corol. Hence if a body be left to float in a fluid, it will turn by its own gravity till the heaviest side is downwards.

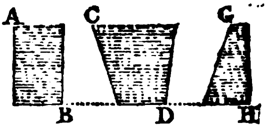
421. *Suppose* HBOPGS *to be a bended funnel or glass containing a fluid, and open at both ends, and* GB *a vertical section through the lowest point; then the two parts of the fluid,* BOPG *and* BHSG, *press equally against that section.*



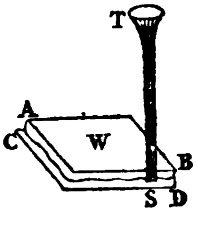
For if the pressure of BHSG the largest body of the fluid were greatest, the other must ascend in consequence of that superior force, but the two surfaces HS and PO will always be in the same horizontal line HO when the fluid is quiescent, whatever be the difference in the diameters HS and PO; that is, the surfaces are constantly at the same vertical height above the lowest part of the fluid (B).

And the pressures are equal in opposite directions at any other section AC; for the pressure of the volume AOPC upwards at AC, is equal to that of AHSC downwards, otherwise the fluid would not rest with the two surfaces HS, PO in the same horizontal line.

Corol. 1. Hence water conveyed to a reservoir by means of a pipe, will rise to the same horizontal level as the spring or head from which it is supplied.

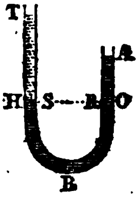


Corol. 2. The pressure of fluids therefore, on equal surfaces alike situated, is the same at equal perpendicular heights, whatever difference there may be in the quantities of the incumbent fluid. Thus suppose AB, CD, GH are three vessels of the same height filled with water, then if the bases are equal, each base will sustain a weight equal to that of the prism of water in the vessel AB whose sides are perpendicular.



Corol. 3. Hence a small quantity of fluid may be made to float or raise a great weight. And on this principle the *hydrostatical paradox* is constructed. Thus AB and CD are two boards connected at the edges by leather so that they can be moved to, and from one another like the top and bottom of a pair of bellows: TS is a small tube open at both ends, that communicates with the inside between the boards. Now when water is poured into the tube at the top T, the board or top AB will be forced upwards by the pressure of the fluid against it on the side, if all the joinings are water-tight. For the force against the board AB is equal to the weight of a prism of water whose base is AB and height equal to the height of the tube TS, and consequently it would raise a weight placed on the top at W equal to the weight of that prism of water. Thus suppose AB and CD are squares, each side being 2 feet, and the length of the tube 6 feet; then $4 \times 6 = 24$ cubic feet is the contents of the prism, and $24 \times 62\frac{1}{2} = 1500$*lb.* its weight. Hence if the internal diameter of the tube be $\frac{1}{4}$ of an inch, about 5 pints of water poured in at the top T will be sufficient to raise AB the upper board $\frac{1}{4}$ of an inch if it be loaded with 13 hundred weight.

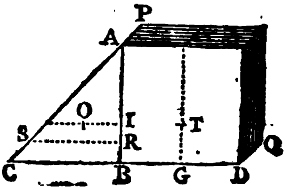
422. *If* TBA *be a bent funnel, or tube, containing two fluids, and* HSRO *the horizontal plane in which they are in contact, the pressure of the denser fluid in the part* AR *is equal to that of the other in* TS.



For if TS and AR were empty, the fluid in HBO would remain quiescent, because the sections (or surfaces) HS and RO are in the same horizontal plane; the fluids in TS and AR must therefore press equally on those sections or surfaces when they are in equilibrio.

Corol. If the sections HS and RO are equal, the perpendicular heights HT and OA will be reciprocally as the specific gravities of the fluids. For in that case the sections, or surfaces, HS and RO are pressed by equal weights, or prisms of the fluids whose heights are HT and OA; and since their bases HS and RO are equal, the specific gravities of the prisms will be inversely as their heights.

423. *Let a cubical vessel* (PD) *whose base is horizontal, be filled with water; then the pressure of the fluid against either side is equal to half its pressure on the base.*



Suppose the square AD to be one of the sides. Produce DB till BC = BA; then the pressure on the bottom, and against one side, will be as the square AD to the triangle

ABC. For since fluids press equally in all directions, and that in proportion to their depths, an indefinitely small column AB, will press at B in the horizontal direction BC with the same force as upon the base at that point B, therefore if the pressure of the column on the base be denoted by the depth AB, its equal pressure in the horizontal direction BC will also be represented by BC; in like manner, the horizontal pressure of the vertical column AR will be denoted by RS which is equal to AR, and so on: consequently all the BC, RS, &c. together, or the area of the triangle ABC will represent the whole horizontal pressure against the side at the line AB; and AB taken BD times (or the area of the square AD) is the pressure on the line BD; but the pressure on the whole base is the weight, or content of the contained fluid, or AD × side of the base (= AD × DQ); and the pressure on either side = triangle ABC × side of the base (= ABC × AP); that is, one pressure is double the other, because the square AD = twice the triangle ABC.

Corol. 1. But if BG (instead of BD) be any other breadth of the vessel, the pressure on the side BP will remain the same as before; the pressure of a fluid therefore, against any upright surface, is equal to half the weight of a prism of the fluid whose base is the surface pressed, and height equal to the perpendicular height of the fluid.

Thus, let the depth of the cubical vessel be 3 *feet*; then $\frac{9 \times 3}{2} \times 62\frac{1}{2}$ $= 843\frac{3}{4}$*lb.* the pressure against one side.

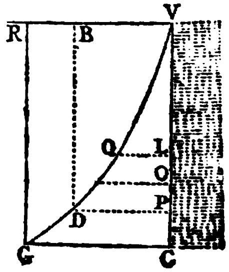
Or suppose the gate or lock supporting water in a canal to be 12 *feet* broad and 10 *feet* deep, and we have $\frac{10 \times 12 \times 10}{2} \times 62\frac{1}{2} = 37500$*lb.* the pressure it sustains.

Corol. 2. Let O be the center of gravity of the triangle ABC; then since the pressures at B, R, &c. in an horizontal direction are denoted by BC, RS, &c, and the whole pressure against AB by the triangle ABC, the center of pressure of the fluid against AB, and center of gravity of the triangle are in the same horizontal line OI, because O is the center of pressure of the triangle; and since AI $= \frac{2}{3}$AB, the centers of pressure and percussion of the upright surface of the fluid BPA coincide, the axis of motion, being supposed at the surface AP. Thus, if BG $=$ GD, and GT $=$ 1 foot or $\frac{1}{3}$ of the height of the vessel, then T is the center of pressure of the fluid against the side AD, and a force $=$ 843$\frac{3}{4}$*lb.* acting perpendicularly against that side at the point T, would just sustain it in equilibrio with the fluid if the side were a loose board.

SCHOLIUM. If ABC be the vertical section of a bank or wall supporting a body of water, the thickness at bottom must exceed the height when its specific gravity and that of the fluid are the same. If it be constructed of brick, which is about twice as heavy as water, the breadth at bottom should therefore be rather more than half the height, supposing it to resist by its own weight only; for if the foundation be sunk below the water, the breadth may be less, but in that case, the thickness must depend on its strength.

Hence in computing the thickness of a wall (Art. 386, &c.) regard should be had to the consistence of the body it is to support, because the center of pressure will vary with its tenacity. Thus the center of pressure of loose earth in an horizontal direction, is usually taken at one third of its depth, but in a fluid it is at two thirds.

424. *Let* CVQG *be the perpendicular section of a wall supporting a fluid; to find the nature of the curve* VQG *when the strength of the wall is every where as the force it sustains, or that it shall be equally liable to break at all depths.*



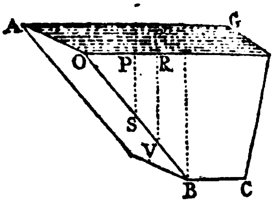
Conceive any depth VP to be divided into innumerable equal parts PO, OL, &c. Then since the pressure of the fluid against any point L is as the depth VL, the number of particles acting by their weight against the wall at that point, will be denoted by VL, and their force at L on the lever PL (considering P as the fulcrum) is PL.VL. In like manner PO.VO will denote the force at O of the particles represented by VO, and so on. Therefore all the PL.VL+PO.VO+ &c. will be the whole force tending to break the wall at P. Hence, if $VP = d$, and any other depth $VL = y$, then all the PL.VL+PO.VO+ &c. will be denoted by all the $(d-y)y$ or $dy - y^2$; now (179) all the $dy - y^2$ is $= \frac{dy^2}{2} - \frac{y^3}{3}$, which (when $y = d$ or VP) becomes $\frac{y^3}{2} - \frac{y^3}{3}$ or $\frac{1}{6}y^3$; therefore the force, or tendency of the fluid to break the wall at any depth, is as $\frac{1}{6}y^3$, or as y^3 the cube of the depth.

And if $b =$ the breadth of the wall, and $x =$ LQ its thickness at the depth y, it follows from Art. 401, &c. that the strength will be as bx^2 or as x^2 (the breadth b being given), that is, the square of the wall's thickness is always as the cube of the corresponding depth of the fluid; or the cubes of the ordinates BD, RG, &c. are as the squares of the abscissas VB, VR, &c. This curve is called a *semicubical parabola:* V is the vertex, and VR the axis.

The thickness of the wall must evidently depend on its strength or tenacity. Let the depth VC $=$ 10 *feet,* thickness at bottom $=$ 5 *feet* $=$ CG,

Then $10^3 : 5^2 :: 6^3 : 5.4$, and the square root of 5.4 is 2.324 *feet,* nearly, the thickness at the depth of 6 *feet,* &c.

425. *If the side* AB *of a vessel containing a fluid, be oblique to the horizon, the pressure it sustains is equal to the weight of a body of the fluid whose magnitude is the surface* AB *multiplied by the perpendicular distance of its center of gravity below the surface of the fluid.*



Suppose the surface AB to consist of an indefinite number of minute surfaces represented by S, V, B, &c. that support the vertical columns PS, RV, &c. then the force or pressure on S, (or content of the column PS) will be denoted by $PS \times S$, and that on V by $RV \times V$, and so on; but all these forces or columns together make up the content of the incumbent fluid.

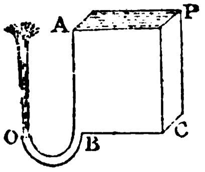
Now if we consider S, V, &c. as a number or system of bodies, it follows from Art. 373, corol. 2, that the aggregate $S + V +$ &c. multiplied by (d) the distance of their common center of gravity from the plane OG, is equal to the sum of the products of those bodies drawn into their respective distances from the same plane; that is, $AB \times d = PS \times S + RV \times V +$ &c. $=$ the pressure on AB, because $S + V +$ &c. $=$ the surface AB. And the same method of reasoning is applicable to any curve surface whatever that supports a fluid.

Let AB be a square whose side is 4 *feet*, and the depth of the vessel $= 3$ *feet*, then the center of gravity of the surface AB is $1\frac{1}{2}$ *feet* below the surface OG; and if the fluid be water, $16 \times 1\frac{1}{2} \times 62\frac{1}{2} = 1500$*lb.* the pressure on the side AB.

And if a hollow sphere be filled with a fluid; then as the center of the sphere is also the center of gravity of the surface, its distance from the highest part of the fluid is equal to the radius, therefore the internal surface multiplied by the radius of the sphere $=$ the pressure; that is, the pressure on the internal surface $= 3$ times the weight of the contained fluid; as in the cubical vessel, Art. 423.

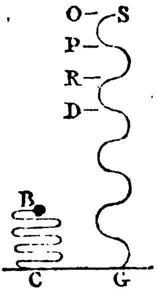
HYDRAULICS.

426. *If a vessel* AC *be full of water, the first or greatest velocity with which it issues through a hole at the bottom, is equal to that which a body would acquire in descending freely from rest through the perpendicular height* AB.



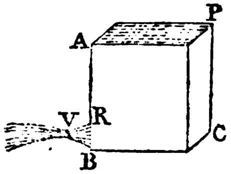
This appears from experiment. For a heavy body projected vertically with the velocity it acquired in descending from a given height, would rise to that same height, provided the motions were made in a non-resisting medium, (317); and when the orifice O of a pipe OB inserted at the bottom is horizontal, and the vessel kept full, the fluid is observed to spout up nearly as high as the surface AP: the stream however, or jet is impeded by the resistance of the air, and therefore its ascent is less than it would be in vacuo.

To determine the velocity from theory, we suppose the weight or pressure of the incumbent column of water to generate that velocity; this force therefore will vary as the perpendicular height of the fluid's surface above the orifice varies, and consequently the force and depth increase uniformly together, like the bending of a spring. Thus let SG be a spring of uniform strength, and suppose that 1*lb.* weight when laid on the end at S will bend it through the space of 1 inch, or from O to P, then it is found that 2*lb.* will bend it twice that space (= OR), and 3*lb.* three times OP or three inches (= OD), and so on; therefore the space through which the spring is bent increases uniformly with the compressing force: hence the velocity of the water may be compared with that of a bent spring when suddenly loosened. Let the spring SG be compressed into the space BC, and a small weight B attached to the end or top B when it is disengaged, then the weight is an uniformly retarding force, and the height to which it will ascend by the force of the spring will be as the square of the initial velocity, as in bodies when projected upwards (317). The same thing however, is evident without supposing a weight at the end B; for since the spring is compresssed by an uniformly accumulated weight or force, it must



regain its unbent position by a motion uniformly retarded. The initial velocities of the spring and effluent water are therefore analogous; and hence the velocity of the water through an orifice at any depth, will be as the square root of that depth.

It is not infered from this determination that water is an elastic fluid. But if the experiments of Mr. Canton are conclusive (Philos. Trans. vol. 53, 54) the whole mass of water belonging to our globe is in a constant state of compression by the weight of the surrounding atmosphere.



But the greatest velocity of the water is at a little distance from the orifice BR. For since the pressure of the fluid in the vessel is in all directions, it is observed that the effluent water converges after it quits the hole, so that when the orifice is circular, it forms the frustum of a cone BRV. Sir I. Newton found that the area of the section at V (which he called the *vena contracta*) is to that of the orifice BR as 1 to $\sqrt{2}$ nearly, and therefore the velocities of the water at BR and V, are as 1 to $\sqrt{2}$. Now it is the velocity of the stream at V which is equal to that acquired by the descent of a heavy body through the whole height AB, consequently the velocity *at* the orifice BR will be nearly the same as a body would acquire in descending through half that height.

Suppose the height BA $= 12$ *feet* $= S$, and let $s = 16\frac{1}{12}$ *feet*, $v = 32\frac{1}{6}$, and $V =$ the velocity at V; then (317) $V = v\sqrt{\frac{S}{s}} = \sqrt{4sS} = \sqrt{772} = 27.78$, &c. *feet*, the velocity per second at V. And $\sqrt{2} : 1 :: 27.78$, &c. : 19.64, &c. *feet* per second, the velocity at the orifice BR.

427. *Let the depth of the vessel be* 12 *feet, and the top and bottom* AP *and* BC *squares, each side being* 5 *feet, to find the time in which it would be exhausted through a circular orifice in the bottom whose diameter* $= \frac{1}{2}$ *an inch, supposing it filled with water.*

Put a = area of the orifice in feet, $b = 25$ feet the area of the top or bottom, $h = 12$ feet the depth, and $s = 16\frac{1}{12}$ feet. Then $\sqrt{2sh}$ will be the first velocity or that with which the water quits the orifice when it begins to run out. And since all horizontal sections of the vessel are equal, it is manifest the surface of the water in the vessel will descend with an uniformly retarded velocity, that is, the velocities will form an infinite arithmetical progression whose first term is $\sqrt{2sh}$, and last term $= o$ the least or last velocity, hence $\frac{\sqrt{2sh}+o}{2} = \sqrt{\frac{sh}{2}}$ feet per second, will be the mean velocity.

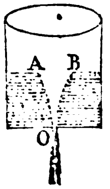
Now the whole length of the cylinder of water that flows through the orifice is $= \frac{hb}{a}$ feet; hence $\sqrt{\frac{sh}{2}} : 1$ *sec.* $:: \frac{hb}{a} : \frac{b}{a}\sqrt{\frac{2h}{s}}$ $= 22397$ *seconds*, nearly, is the time it would require to pass through the orifice with the mean velocity, which therefore is the time of exhaustion.

Corol. 1. And if d = any other height from the bottom, the time of exhaustion will be $\frac{b}{a}\sqrt{\frac{2d}{s}}$; therefore the time of emptying any depth $h - d$ from the top is $\frac{b}{a}\left(\sqrt{\frac{2h}{s}} - \sqrt{\frac{2d}{s}}\right)$.

Corol. 2. The velocities through apertures in the side and bottom are the same at equal depths. And the velocities at different depths are as the square roots of those depths.

Corol. 3. Hence the exhaustion is performed in a less time through a pipe or tube inserted in the bottom; for its lower end becomes the aperture in that case; and since the distance from the surface of the water is thereby increased, the velocity will also be increased.

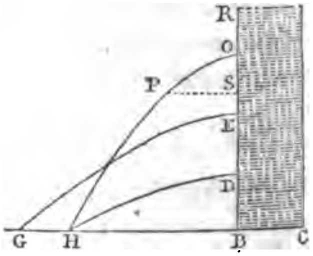
SCHOLIUM. The orifice is here supposed to be completely filled by the effluent water during the time of exhaustion; that however, is not always the case. For when the aperture is large in proportion to the size of the vessel, the surface of the water, soon after the evacuation commences, is observed to move on all sides towards the column directly over the orifice, and this motion becomes more and more perceptible as the surface descends; the water therefore, meeting in different directions, forms an eddy by which means it acquires a spiral motion in descending to the orifice, and the circumjacent water joining in the whirl, produces a funnel-shaped vortex ABO that extends through the bottom at O, the inside ABO being totally void of water. But the difference in the pressure of the air above and below the stream may conduce to the formation of this vortex: for the force of the effluent water diminishes the pressure beneath, on which account the incumbent air following the stream, finds as it were an easier passage; and this appears the more probable, because the velocity of the water in the middle of the effluent column is always greater than that towards the sides, which is retarded by tenacity and friction.



We may therefore conclude that theory and experiment are most likely to agree when the orifice is small.

428. *Suppose* GC *the horizontal plane, and* RC *an upright vessel filled with water; to find the distance* BH *to which it will spout through an orifice* O *in the side of the vessel.*

We consider the effluent water as a projectile discharged horizontally at O with a velocity equal to that which a body would acquire in descending freely from rest through the height RO. And therefore (327, corol. 3, 4) if OS = OR, and SP (perpendicular to RB) = 2OS = 2OR, O will be the *vertex*, S the *focus*, and SP the *semi-parameter* of the *parabola* OPH described by the



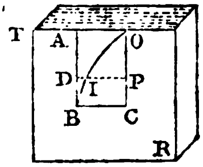
issuing stream. Hence by prop. of the parabola, OS (or OR) : SP^2 (or $4 \times OR^2$) :: OB : $4OR \times OB = BH^2$; consequently $2\sqrt{(OR.OB)} = BH$ the distance.

Corol. 1. Let E be any other aperture, then $BG = 2\sqrt{(ER.EB)}$; therefore the distances BH and BG are as $\sqrt{(OR.OB)}$ and $\sqrt{(ER.EB)}$. And if $BD = OR$, then $DB.DR = OR.OB$, hence it follows that BH is also the distance to which the water would reach through an aperture at D; that is, the horizontal distances are equal through apertures at equal distances from the top and bottom of the vessel.

Corol. 2. If $EB = ER$, the rectangle ER.EB is a maximum, and consequently $2\sqrt{(ER.EB)} = BG$ is also the greatest possible; but when $ER = EB$, BG is $= RB$, therefore the greatest horizontal distance to which the water can spout is equal to the height of the vessel.

If $RB = 13$ feet, OR and DB each $= 4$ feet, then $BH = 2\sqrt{(4 \times 9)} = 12$ feet, supposing the stream is not impeded by the resistance of the air. And when the orifice E is equally distant from the top and bottom, then $BG = BR = 13$ feet.

429. *Let* TR *be a vessel filled with water; to find the quantity that would flow out through the rectangular aperture* ABCO *in a given time, supposing the vessel to be kept constantly full.*



Make O the vertex, and CB the ordinate to the axis CO of the parabola OIB, and let PI be any other ordinate.

Since $\sqrt{OC} : \sqrt{OP}$ :: *veloc. of fluid at* BC : *veloc. at* DP,

And $\sqrt{OC} : \sqrt{OP}$ :: BC : IP, by prop. of the parabola;

Therefore the velocities of the effluent water at the depths OC and OP are as the ordinates BC and IP. Hence, if BC and DP are the perpendicular sections of two indefinitely thin horizontal laminæ of the issuing water, the quantities of the laminæ flowing out in the same time, will be as those sections drawn into the respective velocities, or as $BC \times BC$ and $DP \times PI$; but it will amount to the same thing if the issuing section is denoted by the part PI and its constant velocity by DP or its equal BC, because $PI \times BC = DP \times PI$; consequently if the part PI (or ordinate) were to issue with the velocity of the fluid at the bottom PC, the same quantity would be discharged in the same time as when the whole DP runs out with its real velocity: but the sum of all the ordinates or PI's, &c. together, are equal to, or make up the surface of the parabola; hence if the water ran through the parabolic section OIBC only, but with a velocity every where equal to that at BC, the quantity issuing *would be* the same as that which flows through the whole aperture ABCO.

Put $s = 16\frac{1}{12}$ feet, and let the dimensions of the aperture be also in feet; then $2\sqrt{sOC}$ feet is the velocity per second which a body would acquire in descending freely from O to C, or the velocity of the water at BC; and $OC \times BC$ being the area of the aperture, that of the parabola will be $\frac{2}{3}OC \times BC$ (304); hence $\frac{2}{3}OC \times BC \times 2\sqrt{sOC} =$ the cubic feet per second, the quantity running out.

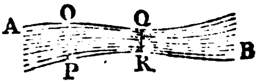
The same conclusion however, is obtained without the parabola by finding the sum of all the velocities from o at the top O, to $2\sqrt{sOC}$ or $2\sqrt{s} \times \sqrt{OC}$ at the bottom C, and dividing that sum by the number of velocities, for the mean velocity; thus, suppose OC to be divided into an infinite number of equal parts OP, &c. then the sum of the infinite progression of square roots $o^{\frac{1}{2}} \times OP^{\frac{1}{2}} + \&c. \ldots OC^{\frac{1}{2}}$, will be $\frac{OC^{\frac{1}{2}+1}}{\frac{1}{2}+1}$, or $\frac{2}{3}OC\sqrt{OC}$ (179), which divided by OC their number, gives $\frac{2}{3}\sqrt{OC}$, therefore $2\sqrt{s} \times \frac{2}{3}\sqrt{OC}$ is the mean velocity, this drawn into $OC \times BC$ the area of the aperture, gives $\frac{2}{3}OC \times BC \times 2\sqrt{sOC}$, as before.

Suppose the depth OC $= 3$ feet, breadth BC $= 2$ feet; then $\frac{2}{3}OC.BC.2\sqrt{sOC} = 55.3$ cubic feet, nearly, the quantity that runs out per second.

But if the velocity at BC is supposed to be equal to that which a body acquires in falling through $\frac{1}{2}OC$, then $\frac{2}{3}OC.BC.\sqrt{2sOC} = 39.3$ cubic feet, the quantity per second. This last result is probably less than would be found by experiment: and on account of the friction against the sides of the aperture, and the oblique motion of the particles in entering it, &c. we may conclude the former, or 55.3 feet, to be greater.

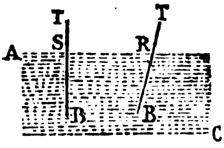
Corol. Hence for the quantity that runs out through an aperture DBCP not reaching to the top; find what would be discharged through ABCO, and ADPO separately, then the difference will be the answer.

430. *Suppose* AB *to represent part of a river or canal whose breadth is unequal; then the velocity of the water at any two places, as* OP *and* QR, *will be reciprocally as the transverse sections of the stream at* OP *and* QR.



For let A and a represent the areas of the sections at OP and QR, and v and V the velocities of the water at those sections, respectively; then Av and aV will denote the quantities that run through OP and QR in the same time; but those quantities are equal, or $Av = aV$, that is $v : V :: a : A$. And the velocities in a pipe are found in the same manner, provided the water fills it at the different sections.

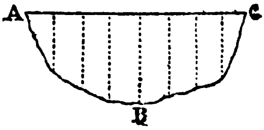
But could we obtain the velocity of the water at the surface, and also the exact dimensions of the section at any particular place (OP), that data would not be sufficient to compute the quantity of fluid that flows along in a given time: for by reason of the friction at the bottom and sides of the canal, the velocity of the water is always greatest at the surface and middle of the stream. The following method however, has been proposed for determining a mean velocity.



Suppose AC is the section of a river or canal in which the water runs from S towards R. And let TB be a small uniform rod of wood loaded in the inside at one end B with a weight sufficient to make it swim with a part ST above the surface. Now this being put into running water, the greater velocity at the surface SR will make it incline towards the direction of the stream, and consequently when it has acquired an equilibrium, it must float along in an oblique position BRT, but the upper part will move slower than the water at the surface, and the lower end B faster than the stream at that depth; and hence we may conclude that the mean velocity of the water will be nearly the same as the velocity of the rod.

The experiment however, should be tried in different parts of the stream, and a mean taken of all the results.

The area of a transverse section may be nearly ascertained by taking a mean of several depths at equal distances across the stream:



Thus, let ABC represent the section; breadth at top AC $= 7\frac{1}{2}$ *feet*, and the depths at 7 equidistant places, 2, $2\frac{6}{12}$, $2\frac{8}{12}$, $2\frac{10}{12}$, $2\frac{4}{12}$, $2\frac{2}{12}$, and $2\frac{1}{12}$ *feet*, respectively; then $\frac{2+2\frac{6}{12}+2\frac{8}{12}+2\frac{10}{12}+2\frac{4}{12}+2\frac{2}{12}+2\frac{1}{12}}{8} = 2\frac{7}{96}$ *feet*, the mean depth, which multiplied by $7\frac{1}{2}$ *feet* the breadth at top, gives $15\frac{35}{64}$ *feet*, the content of the section. (Mensur. *Art.* 271).

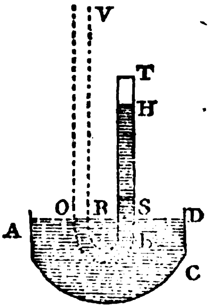
Now supposing the mean velocity of the water is found to be 78 *feet* per minute, we shall have $15\frac{35}{64} \times 78 = 1212\frac{21}{32}$ *cubic feet*, the quantity that flows along in one minute.

The result found in this manner will not, perhaps, be much wide of the truth when the velocity of the water is uniform for a considerable distance. But in small irregular streams and rivulets, other methods must be adopted.

OF PNEUMATICS.

431. **PNEUMATICS** in the ancient schools, and what we understand by Metaphysics, were synonymous: but in the modern philosophy, pneumatics is the science which treats of the weight, pressure, elasticity, &c. of the air.

432. *The atmosphere or mass of air surrounding the globe, is a fluid that presses by its weight on all parts of the earth's surface.*



If a glass tube BT about a yard in length, close at one end T, be filled with mercury and the open end B stopped with the finger while it is immersed in a bason or vessel AC of the same fluid, then on removing the finger from the orifice B, the mercury in the tube will descend into the vessel till the weight of the column SH is equal to the weight of a column of the atmosphere of the same diameter but whose height is equal to the whole height of the atmosphere above the earth's surface. For the end T being close stopped or air-tight, the empty space TH is a vacuum, consequently there is no pressure on the surface of the mercury at H, and hence the column HS is a counterbalance to the weight of the atmosphere acting on the orifice B by its pressure against the surface of the fluid in the vessel; this also appears by letting in the air at the top T, for the mercury will all sink into the vessel; it is therefore kept suspended in the tube by the weight or pressure of the atmosphere. This is called the Torricellian experiment from Torricellius the mathematician and pupil of Galileo, who made the discovery.

The length of the column SH varies from about 28 inches to 31. Suppose the height to be 29½ inches, the tube uniform, and the area of the orifice B = 1 square inch, then the column of mercury SH will be 29½ cubic inches, or 239 ounces; and therefore in a mean state of the air (or when the height of the column SH is $\frac{28+31}{2}$ inches) its pressure upon every square inch of the earth's surface is nearly 15 pounds. So if the orifice at T were stopped with the palm of the hand, a pressure equal to 15*lb*, would be felt on the back.

But the tubes commonly used are 33 or 34 inches long, and the internal diameter about ⅓ of an inch; they are sealed hermetically at the end T; and to the upper part HT is attached a scale of 3 inches divided into 10ths, and subdivided to 100ths, by means of a sliding vernier; the lowest division is numbered 28, and the highest 31, these are inches, and the height of the mercury in the tube above the surface of the mercury in the vessel at bottom is shown at all times in inches and parts of an inch. The tube with its scale, &c. when fixed in a frame is called a BAROMETER.

In fair settled weather the mercury is up at 30 inches, and sometimes higher. When it falls to 29½ a change is usually expected. But it seldom sinks so low as 28 inches, except in very stormy weather.

Mercury is about 14 times heavier than water, therefore if the fluid in the tube TB and vessel AC were water, the height of the column HS would be 29½ × 14 = 413 inches, or about 34 feet, supposing the tube long enough, which is the reason that the piston of a common pump for raising water must work in the barrel or cylinder at a distance less than 34 feet from the surface of the water in the well.

But because the perpendicular pressure of the column of mercury BH and that of the column of air on the whole surface AD are in equilibrio, it may not be readily conceived why the weights of the two columns are unequal. Let us suppose the tube to be lengthened, but bent at the lower end so that its orifice OR is even with the surface of the mercury; also let RV be the column of air having OR for its base, and height RV equal to the height of the atmosphere; then if all the mercury were taken out of the vessel (that in the bent part of the tube excepted) it is evident the equilibrium would still remain, and therefore the *weights* of the columns SH and RV must be equal. For if the columns are cylinders of equal diameters, their weights will be as their pressures.

Since the atmosphere is a fluid variable in weight, it follows that bodies of unequal specific gravities will weigh differently in different states of the air. Thus if a piece of wood is an exact counterpoise to a pound of lead in a nice pair of scales when the mercury in the barometer is at 28 inches, it will not weigh a pound when the mercury stands at 30 inches. For the air is more dense in the latter case by about $\frac{1}{15}$ than in the former, therefore the lighter body or that whose bulk is greatest, will lose more weight or become more buoyant than the heavier, and consequently the lead must preponderate. Hence it has been suggested that advantage may be taken in buying and selling gold and jewels by weighing them in particular states of the weather.

It appears from many experiments that the weight of a cubic foot of air when in a mean state near the earth's surface, is $1\frac{1}{5}$ ounces avoirdupoise, very nearly. Hence the *absolute* weight of a body is what it would weigh in *vacuo*.

433. *If the air was of the same density at all altitudes as at the earth's surface, its height would be between 5 and 6 miles.*

Let RV represent the homogeneous column of air equal in weight to the column of mercury SH which suppose to be $29\frac{1}{2}$ inches. Then (422, corol.) the heights RV, SH will be reciprocally as the specific gravities of air and mercury; therefore $1\frac{1}{5} : 14000 :: 29\frac{1}{2} : 344166$ *inches*, or 5.43 *miles* the height of the column RV. The specific gravities of the two fluids however, will vary if the temperature varies (Schol. *art.* 418). So when the mercury in the barometer stands at 30 inches, and the thermometer at 55 degrees, it has been found by experiments that the specific gravities of air, water, and mercury, are nearly as $1\frac{1}{5}$, 1000, and 13600; hence we shall have, $1\frac{1}{5} : 13600 :: 30 : 340000$ inches, or 5.366 miles, the height of an homogeneous atmosphere whose density would be equal to that of the air at the earth's surface, and weight the same as that of the real atmosphere,

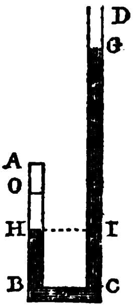
Or taking $1\frac{2}{9}$ for the specific gravity of air, gives 27819 *feet*, or 5.269 miles.

434. *The air is also an elastic fluid that may be condensed, or expanded.*

One of the most simple instruments that shows the air to be elastic, and condensible, is a boy's *popgun*. For by pushing in one pellet with the handle or ramrod, the air that occupied the whole extent of the barrel is condensed till its spring or elastic force is sufficient to propel the other. On this principle the *air-gun* is constructed, only the air is first condensed in the chamber or barrel, and this being suddenly opened by means of a trigger, the air rushes out with great violence against the bullet.

435. *When air is compressed, its density, and elastic force, are proportional to the weight or force that compresses it.*

Let ABCD be a long crooked glass tube, close at the end A, but open at D; the internal diameters of the parts AB and CD being equal. Fill the lower part BC carefully so as to leave the air in BA in its natural state but stop its communication with the external air. This done, pour in more quicksilver till it ascends in the leg BA to H the middle point between B and A; then IG the height of the mercury in the other leg above H will be found equal to the height of the barometer at that time.



The air in HA therefore is condensed into half its former volume by the weight or pressure of an additional atmosphere or the weight of the column of mercury IG which is equal to it; for the air in its natural state was pressed by the weight of the incumbent atmosphere only, but now the compressing force is the column IG together with that of the column of air directly over it; consequently the density is doubled by a double pressure. And since the pressure of the mercury and the action of the air are equal in opposite directions at H, the elastic force of the latter is increased in the same proportion as its density.

Again, if the pressure be doubled by pouring in a sufficient quantity of mercury at D, the mercury will rise to O the middle of HA, and condense the air into the space OA or $\frac{1}{4}$ of BA. In this manner it is found that the density is directly proportional to the compressing force.

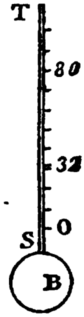
The spaces BA, HA, OA, are reciprocally as the densities or as the compressing forces. It is said however, that after air is condensed into about $\frac{1}{4}$ or $\frac{1}{5}$ of the space it occupied in its natural state, the repulsive or elastic force increases in a greater proportion than the volume diminishes.

Corol. 1. Since the atmosphere at any height from the earth is pressed by the weight of the air above it, the greater that height, the less must be the air's density; thus it is more dense at the foot of a mountain than at the top; and still greater at the bottom of a deep pit than at the earth's surface.

Corol. 2. The elastic force of the air being equal to the force of compression, and (art. 433) 5.366 miles or 340000 inches = 28333 feet the height of an homogeneous atmosphere whose pressure would be equal to that of the real atmosphere, we shall have (art. 426), $\sqrt{(4 \times 16\frac{1}{12} \times 28333)} = 1350$ *feet*, the velocity per second with which common air at the earth's surface would rush into a vacuum.

436. *Heat expands, and cold condenses air.*

Thus, Let a bladder with a little air in it be tied very close, and laid before the fire; then as the inclosed air grows warm, the bladder will distend, and at last burst if the heat be continued. But when the bladder is removed to cool, it will contract again to its former size. The elastic force of the air is therefore increased by heat, and diminished by cold.



It is also found that all fluids, and most solids expand by heat, and contract by cold. And therefore in determining specific gravities, and making experiments with the barometer, it will always be necessary to mark the temperature as shown by the THERMOMETER. This instrument is well known. It consists of a glass bulb B with a small tube ST of the same material fixed in a frame; the bulb and lower part of the tube are usually filled with mercury, but the upper part of the bore should be a perfect vacuum, and the end T hermetically sealed. The degrees of temperature are shown on a graduated scale in the frame. Thus, when it just begins to freeze, the surface of the mercury in the tube will be at the division numbered 32, which therefore is called the freezing point. But when the weather changes to warm, the mercury in the bulb will expand, and consequently rise in the tube, so that in the summer it is sometimes as high or higher than the 80th. division. If the instrument be immersed in boiling water, the mercury will ascend to the 212th. division or degree. And if the heat be sufficient to boil mercury, it will rise to 600 degrees, provided the tube be long enough. This is called *Fahrenheit's* Scale, which was constructed thus: Having observed where the mercury stood in the tube in a very severe frost in Iceland, he marked that point with *o* for the lowest, or the beginning of his divisions; and then determined the other extreme or highest point of his scale by boiling the mercury; the distance between those extreme points he divided into 600 equal parts or degrees; and afterwards observed that the mercury in the tube stood at the 32*d.* division of his scale when water just began to freeze, or snow to melt; and that it rose to the 212th. by the heat of boiling water. It is found however, that the heat of boiling water encreases with the weight of the atmosphere as shewn by the barometer.

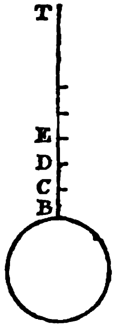
The Thermometers in common use contain about 100 or 120 of the 600 degrees. But for particular purposes the graduations are continued downward from *o*; thus mercury is congealed by the cold at 40° below that point on Fahrenheit's scale. The freezing point on Reaumur's Thermometer is *o*, and that of boiling water 80; sometimes however, this latter point is numbered 110°; hence, by proportion, the different thermometers are easily compared.

The temperate state of the air in England is reckoned at 55°, which is nearly a mean between freezing and summer heat. And to this degree of the thermometer, when the height of the barometer is 30 inches, the table of specific gravities (art. 419) is calculated.

437. The expansions of air, water, and mercury, answering to one degree on Fahrenheit's scale, are about $\frac{1}{435}$, $\frac{1}{6666}$, and $\frac{1}{9600}$ of their bulks, respectively.

$$\text{Expansion of 1 foot of} \left\{ \begin{array}{lcl} \text{Brass rod ....} & = & .0001263 \\ \text{Steel rod.....} & = & .000076 \\ \text{Cast iron....} & = & .000074 \\ \text{Solid glass rod} & = & .0000536 \end{array} \right\} \text{of an inch, by one degree of Fahrenheit.}$$

438. *If vertical distances be taken from the earth's surface in arithmetical progression, the corresponding densities of the atmosphere will decrease in geometrical progression; supposing the force of gravity constant.*



Let BT represent a cylindric column of the atmosphere perpendicularly incumbent upon the surface of the earth at B, and suppose this column to be divided into innumerable thin parts or laminæ of equal heights BC, CD, DE, &c. also let a, b, c, &c. denote their densities in succession from B upwards; then if the magnitude, or the height, of each lamina be represented by m, their weights will be proportional to $m \times a$, $m \times b$, $m \times c$, &c. or ma, mb, mc, &c. (considering the density of each as uniform); and since the densities are as the compressing forces or incumbent weights, the densities at B, C, D, &c. will be respectively

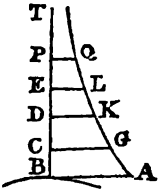
$$\begin{array}{rr} \text{as } ma+mb+mc+md, \text{ \&c.} & \text{or as } a+b+c+d, \text{ \&c.} \\ mb+mc+md, \text{ \&c.} & b+c+d, \text{ \&c.} \\ mc+md, \text{ \&c.} & c+d, \text{ \&c.} \end{array}$$

That is, $a : a+b+c+d$, &c. :: $b : b+c+d$, &c.
or, $a : b+c+d$, &c. :: $b : c+d$, &c. by division.

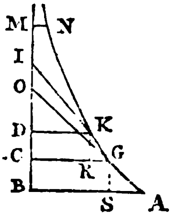
In like manner we get, $b : c+d$, &c. :: $c : d+e$, &c.

therefore by equality
$a : b+c+d$, &c. :: $b : c+d$, &c. :: $c : d+e$, &c.
but $b+c+d$, &c. $c+d$, &c. are as b, c, &c.
hence $a : b :: b : c :: c : d$, &c. That is, a, b, c, &c. are in geometrical progression.

Hence when the heights are in arithmetical progression (the common difference of the terms being m) the corresponding densities a, b, c, &c. will decrease in geometrical progression. Now when an arithmetical series of numbers is adapted to a geometrical one, the former will be analogous to the logarithms of the latter (Arith. 155), it therefore follows that the altitudes increase as the logarithms of the corresponding densities decrease. Or if we make the heights beginning at the surface a descending series, the two progressions will decrease together. Hence any height (BE for example) is proportional to the difference of the logarithms of the densities at B and E, or to the difference of the logarithms of the heights of the mercury in the barometer at B and E, because the densities are measured by those heights. This however, is better explained by means of the *Logarithmic curve ;*



439. Let BT be an indefinite right line perpendicular to the earth's surface at B, and suppose the densities at B, C, D, &c. to be represented by the lines BA, CG, DK, &c. (perpendicular to BT), then a line through the extremities A, G, K, &c. is the logarithmic curve. For if PE, ED, DC, &c. are equal, the abscissas PE, PD, PC, &c. are in arithmetical progression, while the corresponding ordinates (or densities) PQ, EL, DK, &c. are in geometrical progression, which is the property of a logarithmic curve. Thus in Briggs's scale, or the common logarithms, if PQ, EL, DK, &c. are 1, 10, 100, &c. then PE, PD, &c. are 1, 2, &c. (the logarithms of 10, 100, &c.) And since the series 100, 10, 1, $\frac{1}{10}$, &c. may be continued *in infin.* it is evident the curve can never meet BT, which therefore is an assymptote.



440. *In this curve all the subtangents are equal.* Let KI be a tangent at any point K, KD the ordinate at that point, then DI is the subtangent. Suppose DC, CB are indefinitely small but equal parts of the axis; draw the ordinates CG, BA, and let GO be a tangent at G, also make KR, GS perpendicular to CG, BA. Then since CG is a mean proportional between DK and BA, we have

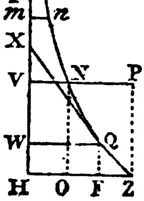
DK : CG :: CG : BA :: (by division) CG—DK (or RG) : BA—CG (or SA),

therefore DK : RG :: CG : SA. But the indefinitely small arcs KG, GA may be considered as right lines coinciding with IK, OG at the points K, G, respectively, and the triangles IKD, KGB; OGS, GAS will be respectively similar, whence

DI : RK :: DK : RG :: CG : SA (by the former proportion) :: CO : SG ;

that is, DI : RK :: CO : SG; but RK = SG, therefore DI = CO. And since the same reasoning will hold good at any point in the curve, all the subtangents must be equal.

441. *Let* NGA (preceding fig.) *and* *n*QZ *be two logarithmic curves,* MN = *mn* = 1, MC, *m*H, *the logarithms of equal numbers or ordinates* CG, HZ; GO, ZX *tangents*; *Then subtang.* CO : *subtang.* HX :: MC : *m*H.



Suppose CD, HW are indefinitely small, but proportional to the logarithms MC, *m*H, or so that MC : CD :: *m*H : HW; and draw DK, WQ parallel to CG, HZ; and KR, QF perpendicular to CG, HZ; then CR and RG are equal to HF and FZ, and the triangles RGK, CGO; FZQ, HZX being respectively similar, we have

CO : RK :: CG : RG :: HZ : FZ :: HX : FQ,

And alternately, CO : HX :: RK (or CD) : FQ (or HW) :: MC : *m*H.

These constant subtangents of the logarithmic curves are called the *moduli* of the systems. In the hyperbolic logarithms the subtangent is 1, and the logarithm of 10 is 2.3025851, now 1 being the logarithm of 10 in Briggs's scale, we have by the last proportion, 2.3025851 : 1 :: 1 : .43429448 the modulus of the common or Briggs's logarithms. Hence if the common logarithm of a number be multiplied by 2.3025851, or divided by its reciprocal .43429448, the result is the hyperbolic logarithm of that number.

442. *If a number be very nearly equal to* 1, *its excess above* 1 *is to its logarithm, as* 1 *to the subtangent.* For let WQ = 1, QX a tangent at Q, HZ the number or ordinate at a small distance from WQ, and WH its logarithm; then the arc QZ and its chord will be nearly equal, and the triangles FQZ, WXQ similar; whence FZ : QF (or WH) :: WQ : WX.

443. *Let* HZ, VN *be any two ordinates,* NO *perpendicular to* HZ, *and* ZX *a tangent at* Z; *then the quadrilinear space* HZNV = OZ *multiplied by the subtangent* HX. For suppose WQ is parallel to, and indefinitely near HZ, and QF perpendicular to HZ; then the triangles HXZ, FQZ may be considered as similar; hence HZ : HX :: FZ : FQ, and HZ.FQ = HX.FZ; but WQ being indefinitely near HZ, the rectangle HZ.FQ may be taken for the area of the quadrilinear HZQW, consequently HX.FZ = the area HZQW; hence if we conceive the quadrilinear HZNV to be composed of an infinite number of elementary quadrilinear spaces HZQW, &c. their areas together will be HX × all the FZ, but all the FZ together = OZ, therefore the quadrilinear HZNV = HX.OZ.

Corol. Hence the infinitely long space contained by the ordinate HZ, assymptote HT, and curve ZNT is = the subtangent HX drawn into the ordinate HZ: For ultimately OZ = HZ.

444. Now let the densities of the air at the earth's surface H, and heights W, V, &c. be denoted by the corresponding ordinates HZ, WQ, VN, &c. respectively; also suppose HV = the height of an uniform atmosphere (Art. 433), and complete the parallelogram VHZP; then the pressure on the surface at H will be proportional to the sum of all the densities or (HZ × subtang. HX) the area of the infinitely long logarithmic space HTZ which is composed of the infinite progression of ordinates. But this is also represented by the parallelogram HP, or HV × density HZ; therefore HV the altitude of an uniform atmosphere is the subtangent of the atmospheric logarithmic; and if ZX be a tangent at Z, the points V and X will coincide.

If therefore HZ be the density at the surface H, and WQ the density at W (which suppose to be the top of a high mountain, for example) then the height (HW) will be proportional to HW the difference of the logarithms of HZ and WQ: But if the curve were actually constructed, its subtangent would be 27819 *feet* (433), and the difference of the logarithms adapted to that subtangent would be the height HW in feet; therefore to find that difference by means of the common logarithms, let D and d be the densities at H and W, then $log.\ D - log.\ d = log.$ of $\frac{D}{d}$; hence to find the *log.* of $\frac{D}{d}$ when the subtang. is 27819, we have $.43429448 : 27819 :: log.$ of $\frac{D}{d} : \frac{27819}{.43429448} \times log.\ \frac{D}{d}$, or $64056 \times log.$ of $\frac{D}{d}$, the *log.* required, or the height HW in *feet*, when the temperature shown by Fahrenheit's thermometer is 55°, and the height of an homogeneous atmosphere = 27819 *feet*, answering to 30 *inches* the height of the mercury in the barometer: But if 29½ *inches* be the mean height of the barometer, we shall get 27355 feet instead of

27819; and taking the mean or 27587 feet for the length of the subtangent, gives $63521 \times log. \frac{D}{d}$ the height in feet, or $10587 \times log. \frac{D}{d}$ the heights in fathoms: But the densities (D, d) are measured by the heights of the barometer at the places of observation H and W, therefore the difference of the common logarithms of those heights $\times$ 10587 is the height of one place above the other in fathoms when the thermometer is at 55°. But as the air expands about $\frac{1}{435}$ of its bulk by 1 degree of Fahrenheit, and the compressing and expanding forces equal, it follows that the result or height must be corrected by adding the 435th. part of itself for every degree which the temperature is above 55°, or subtracting, when the temperature is below. The factor 10587 however, is usually changed to 10000, and the expression for the height given thus, $10000 \times log. \frac{H}{h}$ (where H and h are the heights of the barometer) which is found by reducing the temperature from 55°: Thus, let $l = log. \frac{H}{h}$, then since the height $10587l = 10000l + 587l$, and $\frac{10587l}{435}$ answers to 1 degree, we have $\frac{10587l}{435} : 1 :: 587l : \frac{587 \times 435}{10587} = 24 \frac{1257}{10587}$ degrees, this (neglecting the fraction) when taken from 55°, leaves 31° the temperature in which the expression $10000 \times log. \frac{H}{h}$ gives the height in fathoms; and the result is to be augmented by adding its 435th. part for every degree that the temperature is above 31°; but diminished by subtracting, when it is below.

Examp. 1. What is the air's density at 7 miles above the earth's surface?

7 miles $=$ 6160 fathoms; therefore $10000 \times log. \frac{D}{d}$, or $10000\,(log. D - log. d) = 6160$, whence $log. d = \frac{10000\, log. D - 6160}{10000}$; and assuming D the density at the earth's surface $= 10$, its *log.* is 1, which gives $log. d = .3840$ the *log.* of 2.42, which is nearly $\frac{1}{4}$ of 10; therefore the air is about 4 times rarer at the height of 7 miles than at the surface of the earth.

Examp. 2. Suppose the mercury in the barometer is 29.74 inches high at the foot of a mountain, and 26.41 inches at the summit; what is its height, if the mean temperature be 50°?

29.74 *log.* 1.473341
26.41 *log.* 1.421768
$\frac{H}{h}$ *log.* 0.051573 and 10000 × .051573 = 515.73.

50° — 31° = 19° temperature above 31°; and $\frac{19}{435}$ of 515.73 = 22.53 add
515.73
Height = 538.26 *fath.*

Computing by the formula 10587 × *log.* $\frac{H}{h}$, we have 10587 × .051573 = 546.

55° — 50° = 5° temp. below 55°; and $\frac{5}{435}$ of 546 = 6.28 subtract
546.
Height = 539.72 *fath.*

The difference of the results arises in consequence of neglecting the fraction $\frac{1257}{10557}$.

Examp. 3. If the heights of the barometer at the bottom, and top of a hill, are 29.37 and 26.59 inches, respectively, and the mean temperature 26°; what is the height?

29.37 *log.* 1.467904
26.59 *log.* 1.421718
$\frac{H}{h}$ *log.* 0.043186 and 10000 × .043186 = 431.86.

31° — 26° = 5° temp. below 31°; and $\frac{5}{435}$ of 431.86 = 4.96 subtract
431.86
Height 426.90 *fath.*

445. But on account of the great difference of temperature in low, and elevated situations, several corrections are necessary to make the results from barometrical observations agree with geometrical measurement. Before M. de Luc began his experiments with the barometer, a mean of the two temperatures shown by the thermometer attached to the barometer, and the heights of the mercury in the barometer, at the bottom and top of a hill, were thought sufficient to determine its height. M. de Luc however, found that an additional or detached ther-

mometer was also necessary, (see his *Recherches sur les Modifications de l'Atmosphere*), and this has been confirmed by the experiments of Gen. Roy, and Sir G. Shuckburgh. The formulæ for the height (in fathoms) according to the two latter observers are the following:

Gen. Roy............... $(10000l \mp .468d) \times (1 + (f - 32^\circ) \times .00245)$.

Sir G. Shuckburgh... $(10000l \mp .440d) \times (1 + (f - 32^\circ) \times .00243)$.

Where $l =$ the difference of the logarithms of the heights of the barometer at the two stations,

$d =$ the difference of the degrees shown by Fahrenheit's thermometer attached to the barometer,

$f =$ the mean of the two temperatures shown by the detached thermometer exposed for a few minutes to the open air in the shade at the two stations.

The sign — takes place when the attached thermometer is highest at the lower station, and the sign + when it is the lowest at that station.

Examp. To find the height of a mountain from the following observations taken at the foot, and summit:

	barom.	attached therm.	detached therm.
Lower station.......	29.862	68°	71°
Higher station......	26.137	63°	55°

inches.

Barom. 29.862...*log.* 1.475119 attached therm. 68° detac. ther. 71°

26.137...*log.* 1.417256 63° 55°

diff. $0.057863 = l$. diff. $5 = d$ mean $63 = f$.

By the first formula, $f - 32^\circ = 31^\circ$, and $1 + (31 \times .00245) = 1.07595$

$10000l = 10000 \times .057863 = 578.63$

$.468d = .468 \times 5 = 2.34$ subtr. 2.34

576.29, and $576.29 \times 1.07595 = 620$ *fathoms*, the height.

In computing the height by the formula $10000 \times log. \frac{H}{h}$ (in the preceding article) we take the mean temperature by the detached thermometer;

and correct the barometer for the difference of temperature shown by the attached thermometer: Thus, since mercury expands about $\frac{1}{10000}$ of its bulk by 1 degree of the thermometer, $\frac{5}{10000}$ of 26.137, or .014 of an inch will be the correction for 5°, this added to 26.137 gives 26.151 inches the corrected height where it was coldest.

29.862 *log.* 1.475119   63°—31° = 32° mean temp. above 31°.
26.151 *log.* 1.417488
diff. 0.057631

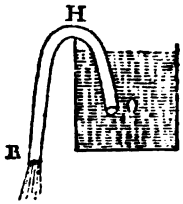
Then $10000 \times .057631 = 576.31$, and $576.31 + \frac{32}{435} \times 576.31 = 618.7$ *fath.* the height.

Ramsden's engraved Table gives the height = 3730 feet, or 621⅔ fathoms. This Table is on a slip of paper 1 foot long, and about 3¼ inches wide; the logarithmic differences from 25 to 31 inches are given to 500ths. of an inch, and the corrections for the thermometers at both stations found by inspection.

Remark. In determining altitudes by the barometer, it is best to make the observations at the upper and lower stations at one and the same time as nearly as can be; but great care must be taken that the two barometers, and also the thermometers, are alike; that is, they should precisely agree when together in all states of the air. It is also necessary that the specific gravity of the mercury be well ascertained, because it is not equally pure in all barometers; which is the principal reason why different results are so frequently obtained from observations made with different barometers at the same stations. Other circumstances however, not generally known, may contribute to such disagreement: thus, Mr. Ramsden proved by experiment that the quicksilver in barometer tubes made of different sorts of glass will be suspended at different heights.

OF THE SYPHON AND PUMPS.

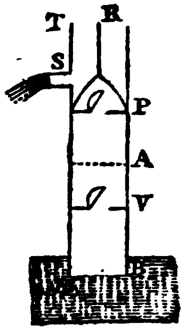
446. **The Syphon** is a bent tube RHO for drawing off liquors: one leg HR is usually made longer than the other, so that when it is resting on the side of a vessel, the outward end R falls lower than O the end immersed.



To use the syphon, fill it with the liquor, and stop both ends while the end O is immersed in the vessel, then if they are opened, the fluid will continue to run out at R as long as that end is lower than the surface of the liquor in the vessel, provided the end O be kept under that surface. For the weight of the column of fluid in HR is greater than that of the column in the other leg, therefore (considering the bend at H as the fulcrum) the former column must descend; and the effluent stream is continued by the constant pressure of the atmosphere on the surface of the liquor in the vessel, which makes it ascend in the leg OH.

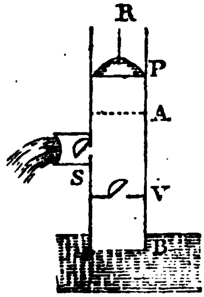
If the vessel contain water, the bent part of the syphon (H) must be less than 33 or 34 feet from the water's surface, because that is the greatest height to which water will ascend by the pressure of the atmosphere.

447. *The common* **sucking Pump.** This is a hollow cylinder or barrel TB containing a fixed valve V, and a piston P moveable up and down by means of a rod R fixed to a handle; in the piston is another valve, and both valves open upwards.



To work the pump. Let PA be that part of the barrel in which the piston moves, and suppose both valves to be shut, and the lower end B immersed in water. Force down the piston, and the air beneath will open its valve, then draw it up, and the valve shuts by the pressure of the air above, by which means the column of air in AP is lifted up or drawn out of the barrel; now the quantity of air which occupied PB being diminished, the air in BV will expand and open the valve V; thus the internal air becomes rarefied, and therefore the external air by its pressure on the surface of the water at B will raise it a little in the barrel. Again, force down the piston, and the air in PV will shut the lower valve, but open the upper one, then by lifting the piston another quantity of air is expelled, and the water in consequence rises higher; thus by continuing the operation, all the air is drawn out of the pump, and the water will ascend above the valve V and be lifted by the piston till it runs out at the spout S.—Water poured into the top of the pump will exclude the external air, should the piston not fit the barrel quite close enough to be air-tight.

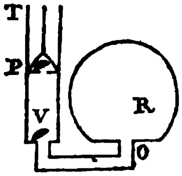
The pressure of the external atmosphere must raise the water above A, which therefore cannot be more than about 32 or 33 feet from the surface at B.



448. **Forcing Pump.** In this the piston P, which is without a valve, works above the spout S where a valve opens outwards. To expel the air (the valves being shut) force down the piston, suppose to A, and a quantity of air equal to that in PA will escape at S by forcing open the valve, then on drawing it up again that valve shuts by the pressure of the external air, and the air in PV being thus diminished, the valve V is opened by the expansion of the air beneath as in the other pump. Now if the piston be again forced down, the lower valve will close and another quantity of air be forced out at S; and since the water rises in the barrel every time the piston is drawn up, it will finally ascend to P (if PB is not more than

about 32 feet) and be forced through the valve at S by the descent of the piston. On this principle the engine for extinguishing fires is constructed.

If the end B of the barrel be closed, and the valves made to open in the contrary directions, or that at V downwards, and the other at S inwards, it becomes a *condenser*. Thus when the piston P is pressed down (suppose to V) the air in PV will be forced into VB; then on lifting the piston, the valve V is shut by the spring of the inclosed air, and the external air will rush in at S and fill the space VP; and by depressing the piston again, another quantity of air is forced in, and so on: in this manner the air in VB will become more and more condensed.



449. *The* AIR PUMP. This is a machine contrived for drawing the air out of a vessel which in experiments, is usually called the *receiver*. The principle is the same as in the common pump: Thus, TV is a barrel in which the piston P (with a valve opening upwards) works perfectly air tight, R is the vessel or receiver, this communicates internally with the barrel by means of a bent tube OV, and at V is a valve that opens upwards as in the common pump. Now when the piston is forced down to V and then drawn up, it lifts out or expels the air in VP which is immediately filled again by the air in the vessel R that expands through the tube OV; in like manner, by depressing and lifting the piston, another quantity of air is drawn out; and if the operation be continued, the air in the receiver may be exhausted till its elastic force is too weak to open the valve at V.

Hence if the capacity of VP, and that of the receiver and tube are given, we can find how much the air is rarefied by any number of lifts or strokes of the piston thus,

Let 1 denote the air in PV, VO, and R together, and $\frac{1}{p}$ that in PV, Then $1-\frac{1}{p}$ or $\frac{p-1}{p}$ is the air left after the first stroke; and since the remainders are successively diminished by the pth. part, we have

$\frac{p-1}{p} - \frac{p-1}{p^2} = \left(\frac{p-1}{p}\right)^2$ the remainder after the 2d. stroke;

$\left(\frac{p-1}{p}\right)^2 - \left(\frac{p-1}{p}\right)^2 \times \frac{1}{p} = \left(\frac{p-1}{p}\right)^3$ .. after the 3d. &c.

that is, the remainders form a descending geometrical progression, the first term being 1, and common ratio $\frac{p-1}{p}$ and therefore the remainder after n strokes will be $\left(\frac{p-1}{p}\right)^n$; but the remainders successively occupy the same space, and consequently the densities are denoted by the terms of the series.

Suppose the capacity of the tube VO and receiver R together is equal to 10 times that of PV the part of the barrel in which the piston works, then PV is $\frac{1}{11}$ of the whole, or $p=11$, and let $n=50$; then $\left(\frac{p-1}{p}\right)^n = \left(\frac{10}{11}\right)^{50}$ $=.008518$ the density of the inclosed air after 50 strokes or lifts of the piston, which is nearly $\frac{1}{117}$ of 1 the first density; so the air is rarefied about 117 times by 50 strokes.

450. Should it be required to determine how many strokes would be necessary to rarefy the air a proposed number of times, let $r=$ that number, then the density will be $\frac{1}{r}$, and we get $\left(\frac{p-1}{p}\right)^n = \frac{1}{r}$, whence $n \times \log. \frac{p-1}{p} = \log. \frac{1}{r}$. Suppose $r=60$, and let $p=11$ (as above), then $n = \frac{\log. \frac{1}{60}}{\log. \frac{10}{11}}$, or (taking the reciprocals) $n = \frac{\log. 60}{\log. \frac{11}{10}} = 43$ nearly, the number of strokes by which the air would be rarefied 60 times.

But a complete air pump is constructed with two barrels; and furnished with various apparatus for different experiments.

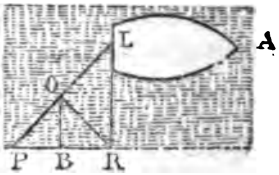
OF THE RESISTANCE, AND THE FORCE OF FLUIDS.

451. *When a body moves in a fluid at rest, the resistance is as the square of the velocity.*

For the resistance is evidently as the velocity of the body drawn into the number of particles it strikes, or if $v =$ the velocity, and $n =$ the number of particles, the resistance will be as nv; but the number of particles struck in any time is as the velocity, therefore substituting v for n gives v^2, that is, the resistance is as the square of the velocity. And since action and reaction are equal, the force of a fluid moving against a body at rest, is as the square of its velocity.

Corol. If the body be a plane moving perpendicularly to its surface p, the resistance will be as pv^2. For the resistance or reaction of the fluid against an indefinitely small part of the plane is as v^2 or $1 \times v^2$, against double that part as $2 \times v^2$, &c. and therefore against the whole plane p, as $p \times v^2$. Also because the number of particles struck in any time is proportional to the density (d) of the fluid, the resistance will be as $d \times p \times v^2$, or dpv^2.

452. *If a plane* PL *move in the direction* PR *or obliquely against a fluid, and* $s =$ *sine of* LPR *the angle of inclination (radius being* 1), *then supposing the resistance in a perpendicular direction to be* dpv^2) (as in the last corol.), *the resistance in the oblique direction is* dpv^2s^3.



Make LR perpendicular to PR. Then the number of particles struck by the plane moving in the perpendicular, and oblique positions, will be as PL to LR, or as *radius* to the *sine* of the angle LPR; hence the expression dpv^2 when reduced in that proportion becomes dpv^2s; for $1 : s :: dpv^2 : \frac{dpv^2s}{1}$ or dpv^2s; therefore supposing LR a plane moving perpendicular to its surface, the resistance would be as dpv^2s. Let this resistance be represented by PR, and draw RO and OB perpendicular to PL and PR, respectively; then since the resistance perpendicular to LR is to the resistance in the direction RO, as *radius* to the *sine* of the angle LRO (= LPR) the inclination of RO to RL (319, corol. 4) we have *rad.* : PR :: *sin.* OPR : RO, or $1 : s :: \frac{dpv^2s}{1} : \frac{dpv^2s^2}{1^2}$, consequently RO will represent the resistance or the reaction of the fluid in a direction perpendicular to the plane PL; and hence by the resolution of forces, RB and BO will respectively be the resistances in the direction of, and perpendicular to the plane's motion, whence $1 : \frac{dpv^2s^2}{1^2}$ (or OR) :: s (or *sin.* BOR) $: \frac{dpv^2s^3}{1^3}$ or dpv^2s^3, (or RB), the resistance to the plane in the direction of its motion.

Corol. 1. Hence the resistances to LR and LP in the direction RP, are as the square of *radius* to the square of the *sine* of LPR, for $1^2 : s^2 :: \frac{dpv^2s}{1} : \frac{dpv^2s^3}{1^3}$.

Corol. 2. Let c = *cosine* of s, or *sine* of the angle BRO; then $1 : dpv^2s^2 :: c : dpv^2s^2c$ or BO, the resistance to the plane in the direction BO or perpendicular to the direction of its motion; this resistance therefore varies as s^2c, or as $(1^2 - c^2)c$, because $s^2 = 1^2 - c^2$.

Hence we may determine what will be the most advantageous angle the rudder of a ship can make with her way to bring her round. Let LP represent the top of the rudder, and LA or PR the direction in which the vessel LA moves; then the resistance to the rudder in a direction at right angles to LA or PR, that is, in the direction BO, must be the greatest possible, or

$(1^2 - c^2)c = 1^2c - c^3$, a *maximum*, which (Art. 406) will be when $c^2 = \frac{1^2}{3}$; whence $c = \sqrt{\frac{1}{3}} = .57735$ the natural *cosine* of 54° 44′ = LPR the angle which the rudder must make with the ship's way to produce the greatest effect in turning her.

And in the same manner it is found that the wind blowing in the direction of the axis of a windmill, will have the greatest effect to turn the sails at the beginning of the motion, when it strikes them in an angle of 54° 44′.

453. *If a fluid moving with a given velocity v, act against a plane in a perpendicular direction, the real or absolute force on the plane is equal to the weight of a column of the fluid whose base is the plane, and height equal to the height through which a heavy body must descend from rest by its own gravity to acquire the velocity v.*

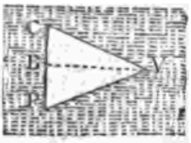
This is manifest from *art.* 426: for the weight or pressure of such a column of the fluid will generate the velocity v; the fluid therefore moving with that velocity must act with a force equal to the weight or pressure which generates it. And if the plane be urged with the velocity v against the fluid at rest, the resistance will be equal to that force of the moving fluid, because action and reaction are equal.

Thus suppose water to move at the rate of 10 feet $(= v)$ per second against a plane surface (p) 1 foot square, and let $d = 62\frac{1}{2}$*lb.* the weight of a cubic foot of water, and $s = 16\frac{1}{12}$ feet; then $\frac{v^2}{4s}$ feet is the altitude due to the velocity v, or the height of the column, $\frac{pv^2}{4s}$ its cubic contents, and the weight or force $= \frac{dpv^2}{4s} = \frac{62\frac{1}{2} \times 1 \times 100}{4 \times 16\frac{1}{12}} = 97$*lb.* nearly, the force of the water against the plane, or the resistance to the plane if it moved perpendicular to its surface through the water at rest.

This force of resistance to the moving plane is not called the retarding force; for if the plane be the face of a body having weight, its momentum, with the same velocity, will vary as its weight, hence the greater that weight, the less will be the retarding force, we therefore divide the resisting force by the weight of the body resisted, and the quotient is the retarding force. So if $w =$ the weight of the body whose plane face is p, then $\frac{dpv^2}{4sw}$ will denote the retarding force, the motive or resisting force being $\frac{dpv^2}{4s}$.

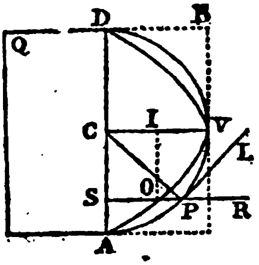
Corol. If the plane be inclined to the direction of its motion in an angle whose *sine* $= s$; then (452) by diminishing $\frac{dpv^2}{4s}$ in the triplicate ratio of *radius* to the *sine* of inclination s, we get $\frac{dpv^2s^3}{1^3 \times 4s}$ or $\frac{dpv^2s^3}{4s}$ the resistance to the plane in the direction of its motion.

454. *If a right cone* CVP *move against a fluid at rest with its vertex foremost in the direction of the axis* BV, *the resistance, to the resistance of a cylinder having an equal base* CP, *and moving also in the direction of its axis, will be as* PB^2 *to* PV^2.



For the same quantity of fluid is struck by the cone and cylinder, and every part of the cone's slant surface is inclined to BV, the direction of its motion, in the angle BVP; therefore (452, corol. 1.) the resistances will be as the square of the *sine* of BVP to the square of *radius*, that is, as PB^2 to PV^2.

455. *When a sphere and cylinder of equal diameters move in a fluid with the same velocity in the direction of the axes, the resistance to the sphere is but half the resistance to the cylinder.*



Let DA be the diameter of the sphere and the cylinder QA, and CV or SR at right angles to DA, the direction of motion. Make LP a tangent to the circle DVA at P, and from the centre C draw CP, and the angles LPR, PCS are equal. Then since the tangent LP and surface of the sphere are struck in the same direction RPS at the point P, the resistance to the sphere at that point, to the resistance at the point S on the face of the cylinder, will be as the square of the *sine* of the angle LPR (or SCP), to the square of *radius*, or (making CP *radius*) as SP^2 to CP^2 (452, corol. 1.). Let CV denote the resistance to any point S on the face of the cylinder, then CP^2 or $CV^2 : SP^2 :: CV : \frac{SP^2}{CV}$ the resistance to the corresponding point P on the sphere, being a third proportional to CV and the *sine* SP. On DA describe the parabola DVA about the axis CV, then $\frac{SP^2}{CV} = SO$:

For let OI be perpendicular to VC;

Then (268) $VC : VI :: CA^2 : IO^2$,

and $VC - VI (= SO) : VC :: CA^2 - IO^2 : CA^2 (= VC^2)$ by division;

But $CA^2 - IO^2 = (CA + IO)(CA - IO) = DS \times SA = SP$, by prop. of the circle,

whence $SQ : VC :: SP^2 : VC^2$, or $\frac{SP^2}{VC} = SO$: therefore the *locus* of the third proportionals SO, &c. or resistances to the semicircular arc DVA, is the parabola DVOA. Now if lines equal to CV are drawn parallel to CV from every point on the end or base of the cylinder, their sum together will denote the resistance to the cylinder, and the corresponding third proportionals on the same points, the resistance to the sphere; but the aggregate sum of the former lines constitute a cylinder BA whose length or height is CV, and all the latter a paraboloid of the same base and height, which in that case is equal to half the cylinder (305), consequently the resistance to the sphere is but half that to the cylinder.

Corol. Hence if p = the area of the great circle of the sphere or base of the cylinder, v = the velocity, $s = 16\frac{1}{12}$ feet, and d = the density or the specific gravity of the fluid; then (451) the resistance to the cylinder will be $\frac{dpv^2}{4s}$ and that to the sphere $\frac{dpv^2}{8s}$.

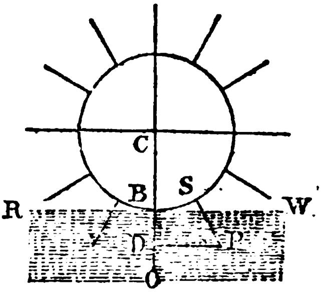
Thus suppose an 18 *lb.* iron-shot to be discharged with a velocity of 1500 feet per second, we have $p = .138526$ of a foot, nearly, $d = 1\frac{1}{5}$ ounces $= \frac{1}{13\frac{1}{3}}$ *lb.* the weight of a cubic foot of air, and $v = 1500$; then $\frac{dpv^2}{8s} = 185$ *lb.* the resistance to the ball.

But the air rushes into empty space with a velocity not greater than between 1300 and 1400 feet per second (435, corol. 2.), the ball therefore moving at the rate of 1500 must leave a vacuum, or the air will cease to act by its pressure on the ball behind during a short space of time, and consequently, in addition to the above result, the ball will be resisted by the weight of a column of the air having a circular base whose diameter is that of the ball. The area of this circle is 19.948 inches, nearly, and allowing 15 *lb.* avoirdupois for the pressure of the atmosphere upon every square inch (432), we get $19.948 \times 15 = 299$ *lb.* which added to 185 *lb.* gives 484 *lb.* the resistance, exclusive of the resistance arising in consequence of the air's being condensed in front by the rapid motion of the ball. And hence it is found that the horizontal ranges are not increased by discharging balls with initial velocities beyond a certain limit.

The theory of Resistances however, and experiment give results considerably different. See Dr. Hutton's Mathematical and Philos. Dictionary, vol. 2, p. 363, &c.

456. *Let a stream of water moving in the direction* WR, *with a velocity* = V, *turn an undershot wheel whose centre*

is C *; then if* v = *the velocity of the pallets or floats* SP, BO, *&c. and* m = *the momentum of the wheel,* m *will vary as* $(V - v)^2 \times v$, *or* $m \propto (V - v)^2 \times v$.



Since the water and floats move with the respective velocities V and v, the former will strike the latter with the relative velocity $V - v$, or the impingent velocity of the water upon the floats is the same as it would be if they were at rest and the water moved with the velocity $V - v$, and hence if the water strike the floats perpendicular to their surfaces, its force (451) will be as $(V - v(^2$.

But the absolute force of the water moving with the velocity $V - v$ is equal to the weight of a column of the fluid having a base equal to the surface of a pallet or float, and whose height is equal to the height through which a heavy body must descend by gravity to acquire that velocity (453), this weight may therefore be represented by the force $(V - v)^2$, whence it follows that the momentum of the wheel will be directly proportional to that weight or force drawn into the velocity v, or to $(V - v)^2 \times v$.

In estimating the force of the stream upon the floats, we take a column of the fluid whose base is equal to the surface of one float only, because the section of the impinging stream which is perpendicular to the direction of its motion is equal to that surface.

Corol. 1. Hence if the velocity of the stream be given, we can determine that of the wheel when its effect is the greatest possible in a given time; for in that case $(V - v)^2 \times v$ must be a maximum; but (406, corol. 1), $(V - v)^2 \times v$ is a maximum when $v = \frac{1}{2}(V - v)$, hence $v = \frac{1}{3}V$, that is, the velocity of the wheel $= \frac{1}{3}$ the velocity of the stream.

Corol. 2. And since the whole force of the impingent water, is to its force on the floats, as V^2 to $(V - v)^2$, or as V^2 to $(\frac{2}{3}V)^2$, that is, as 1 to $\frac{4}{9}$, therefore the resistance of the wheel, including friction, &c. when its effect is a maximum, will be $\frac{4}{9}$ of the resistance which would be just sufficient to counteract or balance the whole force of the water.

SCHOLIUM. The above conclusion is deduced by Maclaurin, Atwood, and other writers who have considered the subject. But it appears from experiments made by Mr. Smeaton. that the maximum effect of a water-wheel is when its velocity, instead of being $\frac{1}{3}$, is nearly equal to $\frac{1}{2}$ the velocity of the fluid. This disagreement it seems, induced a writer (Mr. W. Waring) in the 3d. vol. of the Transactions of the American Philosophical Society, to reject the preceding theory as fallacious, and adopt another founded on the following principle, namely, "that while the stream is invariable, whatever be the velocity of the wheel, the same number of particles or quantity of the fluid, must strike it some where or other in a given time"; and hence it is infered that the force of the stream upon the wheel is in the "simple direct proportion of the relative velocity"; hence (retaining the above notation), the momentum of the wheel will be as $(V - v)\,v$; now if V be a quantity or line divided into two parts $V - v$ and v so that their rectangle $(V - v)\,v$ is a maximum, it follows from *art.* * 239, that the two parts are equal, or $v = \frac{1}{2}V$, that is, the velocity of the wheel $= \frac{1}{2}$ that of the fluid. Which is Mr. Waring's conclusion.

Let the circumference described by the floats be 30 feet, the number of floats $= 30$, and the surface of each $= 1$ foot square; also suppose the velocity of the stream $= 31$ feet per second, and that of the wheel $= 1$ foot; then the quantity of water that strikes a float or floats in one second of time will be 30 cubic feet.

Again, if we conceive the wheel to move round once in a second, the motion of the stream will be $1\frac{1}{30}$ feet, and that of a float 1 foot in $\frac{1}{30}$ of a second, and the quantity of fluid that strikes a float in that time is $1\frac{1}{30} - 1$ or $\frac{1}{30}$ of a cubic foot; now when the lowest float BO is perpendicular to the direction of the stream, four other floats are partly in the water; let us however, suppose that the whole surfaces of 5 floats are struck at the same time, then the quantity of fluid that strikes 1 float in every revolution will be $\frac{1}{30} \times 4$ of a cubic foot, which multiplied by 30 the number of floats gives only 4 cubic feet the quantity of fluid impinging on all the floats in one second of time or during a revolution of the wheel: the difference of the two results is 26 cubic feet: it therefore appears that the number of particles or quantity of fluid which strikes in a given time will depend on the relative velocity; and consequently Mr. Waring's principle (quoted above) must be erroneous. This is also evident from the following consideration, that the velocity of a body, after being struck by running water, may become equal to that of the stream, in which case the body floats without being struck by the fluid.

By increasing the number of pallets on the wheel, the number constantly moving in the water will also be increased, but it does not follow that more surface would be struck, or the velocity of the wheel thereby accelerated; for the number of floats upon a wheel of any diameter may be augmented till its motion in consequence becomes actually diminished.

But in computing the velocity of the wheel according to the common theory, we estimate the force of the water too great by supposing the floats are constantly struck perpendicular to their surfaces, for the direct impact, which can only take place upon a float when in the position BO, is momentary. The particles of water are also conceived to act in succession without impediment, but it is not easy to comprehend how that can actually take place, because the particles in immediate contact with the floats have not room to escape before they are struck by those which follow; the force of the stream therefore seems to be compounded of pressure and percussion. Now these circumstances all tend to shew (what Mr. Smeaton's experiments prove) that the actual force of a stream upon a water-wheel is less than that deduced from the common theory.—We have not considered the effect of friction, because no general rule has yet been devised for that purpose.

ADDITIONAL EXAMPLES

IN

THE APPLICATION OF ALGEBRA, CONIC SECTIONS, MECHANICS, HYDROSTATICS, &c.

1. GIVEN the area of a rectangle $= a$, and the ratio of the sides as m to n; to find the sides.

Ans. $\sqrt{\frac{ma}{n}}$, and $\sqrt{\frac{na}{m}}$.

2. If a rectangle be inscribed in a circle whose diameter $= d$, what are the sides when they have the ratio of m to n?

Ans. $d\sqrt{\frac{m^2}{m^2+n^2}}$, and $d\sqrt{\frac{n^2}{m^2+n^2}}$.

3. If the side of a square be denoted by s, what is the length of that line, drawn from the middle of one of its sides, which divides the area into two parts having the proportion of 2 to 1?

Ans. $\frac{1}{3}s\sqrt{10}$.

4. What is the length of a line drawn from an angle of a rectangle whose sides are S and s that divides the area into two parts having the ratio of 2 to 1?

Ans. $\frac{2}{3}\sqrt{(2\frac{1}{4}S^2 + s^2)}$. Or $\frac{2}{3}\sqrt{(2\frac{1}{4}s^2 + S^2)}$.

5. If the circumference of a circle and the perimeter of a square are equal, which contains the greatest area?

Ans. The circle is to the square as 16 to 12.56637, nearly.

6. If a sphere and cube have equal surfaces, which has the greatest cubic content?

Ans. The sphere to the cube as $\sqrt{6}$ to $\sqrt{3.14159}$ &c.

7. If the three perpendiculars let fall from a point within an equilateral triangle upon the sides, are denoted by a, b, c; what is the side of the triangle?

Ans. $\frac{2a+2b+2c}{\sqrt{3}}$.

8. What plane triangle is that, the natural tangents of whose angles are whole numbers?

Ans.

9. Given the base of a triangle $= b$, the angle opposite the base, and the right line drawn from that angle to bisect the base $= l$; to find the perpendicular.

Ans. $\frac{l^2}{b} - \frac{1}{4}b$.

10. If the base and perpendicular of a triangle are denoted by b and p; what is the side of that inscribed square, one side of which coincides with the base?

Ans. $\frac{bp}{b+p}$.

11. If the sides of a triangle are 28, 25, and 17; what is the side of its greatest inscribed square?

Ans. $10\frac{50}{709}$.

12. If the base and sides of a triangle are denoted by b, S, and s; then what are the expressions for the perpendicular, and segments of the base?

Ans. Perp. $=\sqrt{\left(\frac{S^2+s^2}{2}+\frac{S^2s^2}{2b^2}-\frac{S^4+s^4+b^4}{4b^2}\right)}$.

Greater segm. $=\frac{b^2-s^2+S^2}{2b}$. Less $=\frac{b^2+s^2-S^2}{2b}$.

13. Having observed the elevation of a distant object, I advanced 60 yards directly towards it on a level ground, and then observed the elevation to be the complement of the former to a right angle; advancing 20 yards still nearer, the elevation

now appeared to be just double the first. Hence the height of the object is required?

Ans. 74.16 *yards.*

14. If the radius of a circle be 10, what are the sides of an inscribed triangle when they have the proportion of 2, 3, and 4?

Ans. 9.6824
14.5237
19.3649

15. If $r =$ the radius of a circle, what are the sides of the regular inscribed trigon, tetragon, pentagon, hexagon, octagon, and duodecagon?

Ans. Trigon $r\sqrt{3}$. Octagon $r\sqrt{(2-\sqrt{2})}$.
Tetragon $r\sqrt{2}$. Decagon $r(\frac{1}{2}\sqrt{5}-\frac{1}{2})$.
Pentagon $r\sqrt{(2\frac{1}{2}-\frac{1}{2}\sqrt{5})}$. Duodecagon $r\sqrt{(2-\sqrt{3})}$.
Hexagon r.

16. If the side of a regular trigon, tetragon, pentagon, hexagon, octagon, decagon, and duodecagon be denoted by s, the expressions for their areas are

Trigon $s^2\sqrt{\frac{3}{16}}$. Octagon $s^2(2+\sqrt{8})$.
Tetragon s^2. Decagon $s^2\sqrt{(\frac{125}{4}+\sqrt{\frac{3125}{4}})}$.
Pentagon $s^2\sqrt{(\frac{25}{16}+\sqrt{\frac{125}{64}})}$. Duodecagon $s^2(6+\sqrt{27})$.
Hexagon $s^2\sqrt{\frac{27}{4}}$.

Required the investigations?

17. Let the linear side or side of a face of a tetraedron, hexaedron, octaedron, dodecaedron, and icosaedron (the 5 regular bodies or solids), be denoted by s; then the expressions for their cubic contents will be

Tetraedron $s^3\sqrt{\frac{2}{144}}$. Dodecaedron $s^3\sqrt{\frac{235+\sqrt{55125}}{8}}$.
Hexaedron s^3.
Octaedron $s^3\sqrt{\frac{2}{9}}$. Icosaedron $s^3\sqrt{\frac{175+\sqrt{28125}}{72}}$.

N. B. The Tetraedron has 4 equilateral triangular faces.
Hexaedron or cube, 6 equal square faces.

Octaedron, 8 equal equilateral triangular faces.
Dodecaedron, 12 equal regular pentangular faces.
Icosaedron, 20 equal equilateral triangular faces.

These solids can be inscribed in a sphere.

18. If the length, breadth, and depth of a rectangular parallelopiped are denoted by l, b, and d; what is the diameter of its circumscribing sphere?

Ans. $\sqrt{(l^2 + b^2 + d^2)}$.

19. If the perimeter of a triangle be denoted by p, and the three perpendiculars let fall from the angles upon the opposite sides by a, b, and c; what are the expressions for the sides?

Ans. $\frac{pab}{ab + ac + bc}$. $\frac{pac}{ab + ac + bc}$. $\frac{pbc}{ab + ac + bc}$.

20. In any trapezium, the sum of the squares of the two diagonals is equal to twice the sum of the squares of the two lines joining the middle points of the opposite sides of the trapezium. Required the demonstration?

21. Having the base of a triangle $= b$, the perpendicular upon that side $= p$, and the rectangle of the other two sides $= r$; to find the angle opposite the base.

Ans. $\frac{bp}{r}$ = the *natural sine* of the required angle, (*radius* being 1).

22. Let the base of a triangle $= b$, the tangent of the opposite angle $= t$, and the perpendicular let fall from that angle upon the base $= p$; to find the segments of the base made by that perpendicular.

Ans. $\frac{1}{2}b + \sqrt{\left(\frac{bp}{t} - p^2 + \frac{1}{4}b^2\right)}$, and $\frac{1}{2}b - \sqrt{\left(\frac{bp}{t} - p^2 + \frac{1}{4}b^2\right)}$.

23. If a, b, and c denote the sides of a triangle, what is the radius of its inscribed circle?

Ans. $\sqrt{\frac{(h - a)(h - b)(h - c)}{h}}$, where $h = \frac{1}{2}$ the sum of the three sides.

24. Given the base and vertical angle of a triangle; to find the *locus* of the centre of the inscribed circle.

Ans. The arc of a circle.

25. If the hypothenuse of a right-angled triangle $= h$, and the radius of its inscribed circle $= r$; what are the sides?

Ans. $\frac{1}{2}h + r + \sqrt{(\frac{1}{4}h^2 - hr - r^2)}$, and $\frac{1}{2}h + r - \sqrt{(\frac{1}{4}h^2 - hr - r^2)}$.

26. If $r =$ the radius of three equal circles in contact with each other; what are the radii of the two circles described to touch them internally and externally?

Ans. $2r\sqrt{\frac{1}{3}} - r$, and $2r\sqrt{\frac{1}{3}} + r$.

27. If $r =$ the rectangle made by two lines, and $d =$ the difference of their squares: what are those lines?

Ans. $\sqrt{(\sqrt{(r^2 + \frac{1}{4}d^2)} + \frac{1}{2}d)}$, and $\sqrt{(\sqrt{(r^2 + \frac{1}{4}d^2)} - \frac{1}{2}d)}$.

28. Let the perimeter of a right angled triangle $= p$, and its area $= a$; to find the base and perpendicular.

Ans. $\frac{p^2 + 4a}{4p} \pm \frac{1}{4p}\sqrt{(p^4 - 24p^2a + 16a^2)}$.

29. If the perimeter of a rectangle $= p$, and its diagonal $= d$; what are the sides?

Ans. $\frac{1}{4}p \pm \frac{1}{4}\sqrt{(8d^2 - p^2)}$.

30. If S and s denote the segments of the base made by a perpendicular let fall from the vertical angle of a triangle, and $r =$ the rectangle under the two sides containing that angle; what is the perpendicular?

Ans. $\left(\sqrt{r^2 + \left(\frac{S^2 - s^2}{2}\right)^2} - \frac{S^2 + s^2}{2}\right)^{\frac{1}{2}}$.

31. If the three perpendiculars let fall from the angles of a plane triangle upon the opposite sides, are denoted by a, b, and c; what are the sides?

Ans. $\frac{1}{3}\sqrt{(8a^2 + 8b^2 - 4c^2)}$, $\frac{1}{3}\sqrt{(8a^2 + 8c^2 - 4b^2)}$, & $\frac{1}{3}\sqrt{(8b^2 + 8c^2 - 4a^2)}$.

32. If the three lines drawn from the angles of a plane triangle to bisect the opposite sides, be denoted by a, b, and c; then what are the expressions for the sides of the triangle?

Ans. $\frac{2}{3}\sqrt{(2b^2+2c^2-a^2)}$, $\frac{2}{3}\sqrt{(2a^2+2c^2-b^2)}$, & $\frac{2}{3}\sqrt{(2a^2+2b^2-c^2)}$.

33. To divide an angle whose sine and cosine are denoted by s and c (the radius being 1,) into two parts such, that their sines may have the given ratio of m to n.

Ans. $\frac{sn}{\sqrt{(s^2n^2+(m\pm cn)^2)}}$ = *sine* of less; and $\frac{sm}{\sqrt{(s^2n^2+(m\pm cn)^2)}}$ = *sine* of greater. Where the sign + takes place when the proposed angle is acute, but — when it is obtuse.

34. If the hypothenuse of a right angled triangle $= h$; what are the other sides when the area is the greatest possible?

Ans. Each side $= h\sqrt{\frac{1}{2}}$.

35. What are the sides of the greatest rectangle that can be inscribed in a semicircle, the radius of the circle being denoted by r?

Ans. $r\sqrt{\frac{1}{2}}$, and $r\sqrt{2}$.

36. What is the area of that right angled triangle whose base, perpendicular, and hypothenuse are denoted by x^x, x^{2x}, and x^{3x}, respectively?

Ans. 1.02908, nearly.

37. Given the area of a triangle = 126, the sum of the three sides = 54, and the sum of their squares = 1010. Required the sides?

Ans. 13, 20, and 21.

38. In 4 sides of a regular pentagon traced out for a fortification, stand 4 objects which are found to be at the angular points of a square; now if the side of the pentagon be 180 fathoms, what is the side of that square?

Ans. 190.89 *fath.*

39. A lead ball d inches in diameter is to be cast into two other balls whose diameters are in the given ratio of m to n. Required those diameters.

Ans. $\frac{dm}{(m^3+n^3)^{\frac{1}{3}}}$, and $\frac{dn}{(m^3+n^3)^{\frac{1}{3}}}$.

40. A square piece of ground whose side = 30 yards is to be surrounded by a ditch dug 6 feet deep, and it is necessary that the earth thrown out should be sufficient to raise the interior surface 4 feet higher than the present level; now what must be the breadth of the ditch at bottom, supposing it the same all round, when the slope on each side is 45°, and the inner slope continued up to the new made surface?

Ans. 5.488 *feet.*

41. To determine the height of a hill we observed the elevations of an object on its summit at three stations A, B, and C, in the same horizontal right line, and found them to be 2° 48′¼, 3° 39′¾, and 7° 47′, respectively; the distance from A to B was 900 yards, and that from B to C 750 yards. Hence the height is required?

Ans. 171 *yards*, nearly.

42. Three detachments of foot having orders to occupy a certain post, begin their march from three towns, A, B, and C, at 6 o'clock in the morning; the detachment from A march 4 miles per hour, that from B march 3 miles per hour, and the other from C march 2 miles per hour, and they all arrive at the place of destination exactly at the same time, which was between 10 and 11 o'clock; now the distance from A to B was 15 miles, from B to C 8 miles, and from A to C 20 miles. Hence the distances from the post to the three towns are required.

Ans. 17.6858 *miles* from A.
13.2643 from B.
8.8429 from C.

43. In the course of a survey, at a station on the top of a hill, we took the depressions of three objects, A, B, and C, which were nearly on the same horizontal level, and found them to be $4^\circ\ 52'$, $4^\circ\ 30'\frac{1}{2}$, $7^\circ\ 13'\frac{3}{4}$, respectively; now the distance from A to B was 4 miles, from B to C $3\frac{1}{2}$ miles, and from C to A 3 miles. Hence the perpendicular height of the hill is required?

Ans. 329 *yards*, nearly.

44. If the axes of an ellipse be 60 and 80; what are the lengths of two conjugate diameters, the longest of which makes an angle of 20° with the transverse axis?

Ans. 68.7861, and 79.1737.

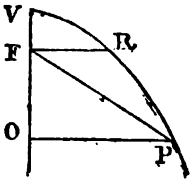
45. Let the axes of an ellipse be 60 and 100 inches; to find the radius of a circle described to touch the curve when its centre is in the transverse axis at the distance of 16 inches from that of the ellipse.

Ans. 27.49545 &c. *inches*.

46. Let VO be the axis of any conic section VRP, F the focus, and FR, OP, two ordinates at right angles to VO; then

$$FP = VF + \frac{(FR - FV)\ VO}{VF}.$$

Required the investigation?



47. If the axes of an ellipse be 80 and 60 yards; what are the areas of the two segments into which it is divided by a line drawn parallel to the conjugate axis at the distance of 10 yards from the centre?

Ans. 1291.27 and 2478,65 *yards*, nearly,

48. If the base of a triangle be given, and also the sum of the squares of the other two sides; what is the *locus* of the vertex of the triangle?

Ans. The arc of a circle,

49. In a triangle, if the base, and the difference of the other

two sides, are given; what is the *locus* of the vertex?

Ans. An hyperbola.

50. Let the base, and the difference of the two angles at the base of a triangle, be given; required the *locus* of the vertical angle in that case?

Ans An hyperbola.

51. Suppose a person, the height of whose eye is 5 feet 6 inches, while standing on a level floor, holds a 9 *lb.* iron shot in his hand so, that its centre is 15 inches from the eye, and 4 feet 9 inches from the floor; now how much of the floor's surface is hid by the shot from the eye; the shot's diameter being 4 inches?

Ans. 8.36344 *feet* square.

52. A heavy body was observed to descend freely from rest, by its own gravity, from the top of a tower to the bottom in 4½ seconds of time; required the tower's height?

Ans. 72¼ *feet.*

53. From what height must a heavy body descend by its own weight to acquire a velocity of 1000 feet per second, supposing the air to be without resistance?

Ans. 15625 *feet.*

54. Suppose a heavy body to fall from the height of a mile above the earth's surface, with what velocity would it strike the ground, and what would be the time of descent?

Ans. Velocity = 582.8 *feet* per second.
Time = 18.12 *sec.* nearly.

55. If a heavy body descend ⅓ of the whole distance fallen in the last second of time; at what height did it commence its motion?

Ans. 477.6 *feet,* nearly.

56. If a ball with a velocity of 1000 feet per second, enters a block of wood to the depth of 10 inches, what will be the velocity of the ball when the penetration is 16 inches, supposing the resistance of the wood to be uniform?

Ans. 1265 *feet* per second.

57. A quiescent body *C* is struck at the same instant of time by two other bodies *A* and *B* with forces that would separately carry it forward in the directions AC and BC at the rate of 15 and 10 feet per second, respectively; required the velocity and direction of *C* after the impact, if the directions of *A* and *B* form an angle at C of 70°?

Ans. Velocity = 20.7 *feet* per second.
Direction 27° $1\frac{2}{3}'$ with that of *A*, or 42° $58'\frac{1}{3}$ with the direction of *B*.

58. Let the directions of *A* and *B*, and the force of *A* continue as before, and suppose after the impact that *C* moves with a velocity of 30 feet per second; required its direction, and the velocity which *B* would communicate alone?

Ans. Direction 28° $1'\frac{1}{2}$ with that of *B*, or 41° $58'\frac{1}{2}$ with *A*.
Velocity which *B* would produce alone = 21.36 feet per second.

59. If the velocities communicated to *C* by *A* and *B* when acting separately, and together, are respectively as 5, 4, and 3; what is the angle formed by the two directions in which *A* and *B* move when they act together?

Ans. 143° $7'\frac{4}{5}$.

N. B. In this and the two preceding examples, the bodies are supposed to be globular, and the points of impact in the lines joining the centres.

60. Suppose the weight *C*, *art.* 320, corol. 2, to be 60 *lb.* what are the tensions of the cords AB, GB, and BC, or the forces with which they are stretched?

Ans. 48, 36 and 60 *lbs.*

61. Let a ring of metal weighing 8*lb.* slide freely on a string 5 feet long whose ends are fastened to two tacks 3 feet asunder in a line making an angle with the horizon of 45°; to find the stress or force on each tack when the ring rests in equilibro.

Ans. 4.415 *lb.* on each tack.

62. Suppose the ends of a thread 10 feet long be fastened to two tacks in the same horizontal line, at the distance of 6 feet; where must two weights, the one 3, and the other 5 ounces, be fixed to the thread, so as to hang at rest in the same horizontal line at the distance of three feet from the level of the tacks?

Ans. At 3.1479, and 3.395 *feet* from the ends of the thread.

63. Suppose a 12*lb.* shot moving with a velocity of 1000 feet per second, to meet another of 9 *lb.* whose motion in an opposite direction is at the rate of 1200 feet per second; what is the velocity after congress, if the balls are non-elastic?

Ans. 57$\frac{1}{7}$ *feet* per second.

64. With what velocity must a 6 *lb.* shot meet another of 24 *lb.* that is moving at the rate of 400 feet per second, so as just to stop it; the balls being non-elastic, and the stroke in the direction of the centres?

Ans. 1600 *feet* per second.

65. A body at rest, but not fixed, when struck by a musket bullet weighing 1 ounce, moved with a velocity of 6 feet per second; now if the body weighed 10 *lb.* what was the velocity of the bullet?

Ans. 966 *feet* per second, if the bullet and body were non-elastic.

66. With what velocity will a 32 pounder recoil when discharged horizontally, if the gun and carriage together are 63 hundred weight, and the initial velocity of the ball = 1300 *feet* per second of time?

Ans. 65 *inches* per second, nearly.

67. Required the ratio of the masses of two elastic balls *A* and *B*, so that *A* striking *B* at rest shall lose $\frac{1}{4}$ of its velocity?

Ans. As 7 to 1.

68. Suppose two cannon shot, one 18 *lb.* the other 12 *lb.* when moving in the same plane, to strike one another in an angle of 80° with the respective velocities of 800 and 1000 feet per second; requuired the velocities and directions after the impact, if the balls are non-elastic.

Ans. Velocity of the greater shot 615 *feet* per second:
of the less 768.
Change in the direction of the greater shot $38^\circ\ 18'\frac{1}{4}$.
in the less $43^\circ\ 45'\frac{2}{3}$.

N. B. The tangent to the two balls at the point of impact is supposed to bisect the given angle 80°.

69. If a cannon ball be discharged from the top of a tower 80 feet high, with an initial velocity of 1500 feet per second, at what distance from the tower will it strike the ground, the elevation of the piece above the horizontal line being 20° 5′, and the air supposed to be without resistance?

Ans. 45335 *feet.*

70. Required the elevation of a mortar to hit an object distant 7333 feet on a plane depressed 11°; the greatest horizontal range being 8190 feet?

Ans. $63^\circ\ 23'\frac{1}{2}$, or $15^\circ\ 36'\frac{1}{2}$.

71. If the horizontal range of a shell be 2000 feet when projected at an elevation of 30°; what is the time of flight, and the greatest height to which the shell ascends?

Ans. Height $= 287\frac{1}{2}$ *feet.* Time $= 8\frac{1}{2}$ *sec.*

72. If the impetus be 4000 feet, what must be the elevation of a mortar to hit an object whose distance on the horizontal plane is 5600 feet, and height above that plane 812 feet?

Ans. 65° 45′, or 32° 30′.

73. A shell being thrown from a mortar at an elevation of 30°, the report of its fall on the horizontal plane was heard at the mortar just 20 seconds after the explosion. Hence the range is required?

Ans. 6033 *feet.*

74. The random of a piece on the plane of the horizon with a given charge of powder at an elevation of 30° being 1500 yards; to find the elevation when planted at 44 yards above the level of the horizon, so that the ball may fall at the greatest distance possible.

Ans. $44° 17'\frac{2}{5}$.

75. A shell discharged at an elevation of 45° struck an object 50 yards above the horizon; required the distance of the object from the mortar, the horizontal range of the shell being 2000 yards.

Ans. 700, or 1300 *yards.*

76. A tower built on level ground is 65 feet high, now at what distance must I stand with a musket to hit an object on the top with the greatest force, the musket being held 5 feet above the ground?

Ans. At 60 *feet* from the tower.

77. In what time would a heavy body descending freely on a plane inclined to the horizon in an angle of 40° acquire a velocity of 100 feet per second?

Ans. 4.337 *seconds.*

78. A body descending freely by its own weight on an inclined plane whose length is 484 feet, descends 123 feet in the last second of time; required the plane's inclination to the horizon?

Ans. $34° 1'\frac{2}{3}$.

79. A cylinder was observed to *roll* down a plane 400 feet long in 16 seconds of time; required the plane's inclination to the horizon, the cylinder having descended by its own gravity?

Ans. $8° 21'\frac{1}{4}$.

80. If one end of a beam 20 feet long be 6 feet higher than the other end, what force acting in direction of the beam would keep a weight of one ton laid upon it from sliding down, supposing the friction between the weight and beam is equal to half the necessary force?

Ans. A force equal to 3 hundred weight.

81. If a man can draw a weight of 84 *lb.* up the side of a perpendicular wall 10 feet high, what weight will he be able to raise along a plank 20 feet long laid aslope from the top of the wall, the resistance from friction on the plank being equal to $\frac{1}{3}$ of the weight so raised?

Ans. 126 *lb.*

82. Two planes HV and OV whose lengths are 5 and 3 feet, respectively, meet at V above the horizontal line HO, and two weights *A* and *B* connected by a string passing over a pulley at V, are in equilibrio on the planes; now the pressure of *A* upon the plane HV is double that of *B* against OV. Hence the height of V above the horizon HO is required?

Ans. $\sqrt{\frac{11}{3}}$ *feet.*

83. Suppose two weights, one of 6, the other of 2 pounds, to be suspended upon a pin by means of a string, to determine how far the greater will descend, and the other ascend in 1 second of time, neglecting the friction on the pin.

Ans. $8\frac{1}{24}$ *feet.*

84. If a pendulum vibrating in an arc of 24° be 40 inches long, what is its velocity at the lowest point, supposing a body descends $16\frac{1}{12}$ feet in the first second of time?

Ans. 11.68 *inches* per second.

85. If the distance from the point of suspension to the centre of oscillation of a pendulum be 3 inches, how many vibrations will it perform in a minute, in the latitude of London?

Ans. 217 nearly.

86. What must be the length of a pendulum to vibrate only 40 times in a minute?

Ans. $88\frac{1}{8}$ *inches.*

87. If a slender uniform rod 4 feet in length, be suspended at one end, and made to vibrate in small arcs, how many times will it oscillate in a minute?

Ans. 66.38 nearly.

88. What weight can a man raise with a handspike or iron crow 8 feet long, if the fulcrum or prop is 5 inches from one end, and he presses with a force equal to 150 *lb.* at the other?

Ans. 2730 *lb.*

89. If one arm of a steel-yard is 3 inches, what must be the length of the other that a counterpoise of 10 *lb.* may be sufficient to weigh a hundred weight, supposing the weight of the instrument itself is not considered in the account?

Ans. 33.6 *inches.*

90. The cylinder or axle over a common draw-well is 3 inches in diameter, the rope $\frac{3}{4}$ of an inch in diameter, and the handle describes a circle 30 inches in diameter; now what weight can a man draw up who acts with a force equal to 40 *lb*?

Ans. 320 *lb.*

91. Which is drawn with the least force on a rough uneven road, a carriage having wheels of 3 feet in diameter, or one with wheels that are 5 feet in diameter?

Ans. The advantage in favour of the greater wheels is as 5 to 3.

92. If the screw of a press be turned with a lever 7 feet long, and the threads of the screw are 1 inch asunder; what is the force of the press when the power at the end of the lever is = 100 *lb.* supposing the screw to act without friction?

Ans. 52779 *lb.*

93. A man with a combination of pullies raises a heavy body $1\frac{1}{2}$ inches at every pull which draws the rope 36 inches; now what is the weight of the body if he pulls with a force equal to 80 *lb*?

Ans. 1920 *lb.*

94. A barrel of gunpowder weighed 81 *lb.* in one scale, but when put into the opposite scale, it was found to weigh only 78 *lb.* $12\frac{1}{4}$ *oz.* Hence the true weight is required?

Ans. 79 *lb.* 14 *oz.*

95. Three inches from one end of a cylindrical pole is hung a weight of 30 *lb.* the pole is 8 feet long, and weighs 10 *lb.* now how far from that end must I place the pole on my shoulder to carry the weight with the most ease?

Ans. $14\frac{1}{4}$ *inches.*

96. What must be the length of a cylinder, the diameter of whose base is a yard, so that it may just stand by its own weight on sloping ground which rises 1 yard in 10?

Ans. $\sqrt{99}$ *yards.*

97. To find the centre of gravity of a quadrangular board of uniform thickness, two adjacent sides being 17 inches each, the other two 14 inches each, and the shortest diagonal = 16 inches.

Ans. 13.8297 *inches* from the sharpest corner.

98. A beam of timber 20 feet long is to be supported in an horizontal position by two props; the ends of the beam are squares whose sides are 2, and 3 feet, respectively; now if one prop stands 4 feet from the greatest end, at what distant from the less end must the other be to bear an equal weight?

Ans. $6\frac{12}{19}$ *feet.*

99. To determine the weight of a tapering beam of timber 30 feet long, we found that it rested in an horizontal position on a prop or fulcrum 16 feet from the less end, but when the middle

of the beam was brought over the prop, it required the weight of a man, which was 200 *lb.* at the less end to keep it in equilibrio. Hence the weight is required?

Ans. 3000 *lb.*

100. The weight of a ladder 20 feet long is 70 *lb.* and its centre of gravity 11 feet from the less end; now what weight will a man sustain in raising this ladder when he pushes directly against it at the distance of 7 feet from the greater end, and his hands are 5 feet above the ground?

Ans. 63 *lb.* nearly.

101. If the quantity of matter in the moon, be to that of the earth, as 1 to 39, and the distance of their centres 240000 miles; where is their common centre of gravity?

Ans. 6000 *miles* from the earth's centre.

102. Supposing the *data* as in the last question, to find the distance from the moon in the line joining the centres, where a body would be equally attracted by the earth and moon; the force of attraction in bodies being directly as the quantities of matter, and inversely as the squares of the distances from the centres.

Ans. $\frac{240000}{1+\sqrt{39}} = 33126\frac{1}{2}$ *miles*, nearly.

103. If two fires, one giving 4 times the heat of the other, are 6 yards asunder; where must I stand directly between them to be heated on both sides alike; the heat being inversely as the square of the distance?

Ans. 2 *yards* from the less fire, or 4 from the greater.

104. To what height above the earth's surface should a body be carried to lose $\frac{1}{10}$ of its weight; the earth's radius being 3970 miles, and the force of gravity inversely as the square of the distance from its centre?

Ans. $214\frac{1}{4}$ *miles.*

105. How far beneath the surface should the body be to lose $\frac{1}{10}$ of its weight, the force of gravity, in that case, being directly as the distance from the centre?

Ans. 397 *miles.*

106. If a line $= l$, be drawn from a point P to the centre of a circle whose diameter $= d$, and they revolve together about the point P, the circle, moving perpendicular to its plane, will generate a ring (like the ring of an anchor); required its solid content?

Ans. Let $c = 3.1416$, then $\frac{1}{2}lc^2d^2 =$ the content.

107. Suppose the point P to be at the circumference of the circle, or let the circle revolve about a tangent to its circumference as a fixed axis, then what is the content of the generated solid?

Ans. $\frac{1}{4}c^2d^3$.

108. Let a semicircle revolve about the tangent parallel to its diameter; required the content of the solid in that case.

Ans. $d^3\left(\frac{1}{8}c^2 - \frac{1}{6}c\right)$.

109. If a spar of wood 2 inches broad, and $\frac{1}{2}$ an inch thick, will bear 50 *lb.* with its broadest side horizontal; what would it support when that side is vertical?

Ans. 200 *lb.*

110. A spar of oak when resting on its ends in an horizontal position will bear 200 *lb.* at a certain point; now what weight will it support (at the same point) when it is inclined to the horizon in an angle of 60°?

Ans. 400 *lb.*

111. Let $a =$ the magnitude of a mass or ingredient, and A $=$ its weight.
$b =$ the magnitude of another ingredient, $B =$ its weight.
$m =$ the magnitude of any mixture or mass of both, and $M =$ its weight:

Then $\frac{Mba - Bma}{Ab - Ba}$, and $\frac{Abm - Mba}{Ab - Ba}$ will be the respective magnitudes of the ingredients in the compound. Required the investigation?

112. If n = the cubic feet in a mass of metal whose specific gravity or the number of ounces in a cubic foot = m, the specific gravity of wood = d, and the specific gravity of water = w; then $\frac{(m - w)\,n}{w - d}$ = the cubic feet of wood that will just float the metal. For example, 179.187 cubic feet of deal will float a cast-iron cannon of 52 hundred weight, in fresh water. Required the investigation?

113. How many empty 54 gallon casks (beer measure) when immersed in sea water, would float a brass cannon weighing 18 hundred weight, supposing the casks are water-tight, made of oak, and the weight of each = 50 *lb*?

Ans. 3.216, or 3 fifty-four gallon casks, and another that holds about 11 gallons.

114. If 4 *lb.* of fine silver, 6 *lb.* of copper, and 9 *lb.* of tin, are melted together, what is the specific gravity of the composion?

Ans. $8644\frac{8}{19}$.

115. If the weight of a shell $12\frac{1}{4}$ inches in diameter, be 198 *lb.* what is its thickness?

Ans. 2.02 *inches.*

116. If a sphere of wood 9 inches in diameter sinks, by its own gravity, 6 inches in fresh water; what is its weight, and specific gravity?

Ans. Weight 10 *lb.* 3.6 *ounces.*
Specific gravity 741.

117. To what depth would a globe of elm, whose diameter is 10 inches, sink by its own weight in fresh water?

Ans. 5.6707 *inches.*

118. Suppose the outward dimensions of a pontoon are

$$\left.\begin{array}{r}\text{Length at top} = 26\\ \text{at bottom} = 23\end{array}\right\}\text{feet.}\qquad \left.\begin{array}{r}\text{Breadth} = 2\tfrac{1}{4}\\ \text{Depth} = 2\tfrac{1}{3}\end{array}\right\}\text{feet.}$$

What weight will sink it 2 feet in fresh water?

Ans. 6160 *lb.* including the weight of the pontoon.

119. If a cube of wood floating in sea water, be $\frac{3}{4}$ wet, and it sinks $\frac{3}{10}$ of an inch deeper in fresh water; what is its magnitude, and specific gravity?

Ans. Side of the cube = $13\frac{1}{3}$ *inches.*
Specific gravity = $772\frac{1}{2}$.

120. It has been found by experiment that the mean specific gravity of human bodies when alive, is about 891 (that of fresh water being 1000); hence it is required to determine how many pounds of cork would be sufficient to float a person weighing 180 *lb.* with only $\frac{3}{4}$ of his body in water?

Ans. 9 *lb.*

121. Suppose a spherical balloon 27 feet in diameter can just raise 600 *lb.* including the balloon and its apparatus; now if that weight (600 *lb.*) be of the same specific gravity as water, and the specific gravity of common air = $1\frac{2}{9}$; it is required to determine the specific gravity of the inclosed *gas* or inflammable air?

Ans. $\frac{291}{1000}$ nearly, or about $4\frac{1}{5}$ times lighter than common air.

122. If the diameter of a cylindrical vessel be 20 inches; required its depth, that when filled with a fluid, the pressure on the bottom and sides may be equal to each other?

Ans. 10 *inches.*

123. In what time would a ditch whose breadth at top = 16 feet, at bottom 8 feet, depth = 6 feet, and length = 100 yards, be filled with water through a rectangular opening 1 foot deep, and 2 feet wide, cut in the bank of a river; the top of the cut or

opening being on a level with the surface of the water in the river?

Ans. 47 *min.* 36 *sec.*

124. To find the whole force of water moving with a velocity $= \frac{1}{100}$ of a foot per second of time, against a rectangular flood-gate standing perpendicular to the horizon, whose breadth $= 12$, and depth $= 6$ feet.

Ans. $13500\frac{27}{3860}$ *lb.*

125. Suppose a musket barrel $\frac{3}{4}$ of an inch in the bore to contain water, and let the water be forced down by means of a sponge at the end of the ramrod with a pressure $= 50$ *lb.* now (neglecting the resistance of the air) with what velocity will the water issue through the touch hole, if the sponge be air-tight, and the velocity of issuing water equal to that acquired by the free descent of a heavy body through the whole distance from the surface to the aperture.

Ans. 115 *feet* per second.

126. If an empty common glass bottle be corked and sunk in the sea 60 fathoms deep; with what force is the cork pressed by the water if the mouth of the bottle be $\frac{7}{8}$ of an inch in diameter?

Ans. $96\frac{3}{4}$ *lb.*

127. A glass cylindrical vessel whose depth $= 2$ feet, was sunk in the ocean with the open end downwards till the water rose 21 inches within the vessel; hence the depth to which it was sunk is required, supposing the pressure of the atmosphere to be $14\frac{3}{4}$ *lb.* on a square inch?

Ans. $38\frac{1}{2}$ *fathoms.*

128. If a man can push with a force $= 100$ *lb.* how far will he be able to introduce a sponge into a piece of ordnance whose length is 7 feet, and calibre 4 inches, when the barometer stands at 30 inches; the vent or touch-hole being stopped, and the sponge without windage?

Ans. $29\frac{2}{5}$ *inches*, nearly.

129. If a diving bell in the form of a cone, having the internal diameter of its base = 8 feet, and perpendicular height = 12 feet, be sunk in the sea to the depth of 13 fathoms; to what height will the water rise in the inside, and how much is the inclosed air condensed; the pressure of the atmosphere being $14\frac{3}{4}$ *lb.* on a square inch?

Ans. 4 *feet*, ascent of the water.
Density of the internal air, to that at the earth's surface as $3\frac{3}{8}$ to 1.

130. According to Humboldt, the height of the mountain Chimboraço one of the Cordilleras in South America, is 19600 French or 20889 English feet; now how much rarer is the air at the top of the mountain than at the bottom, supposing the height of the barometer at the latter situation to be 30 inches, and the specific gravities of air and quicksilver, $1\frac{2}{9}$ and 13600, respectively?

Ans. Nearly $2\frac{8}{25}$ times rarer.

131. To what height would the balloon in Examp. 121 ascend, if the attached weight and balloon together (exclusive of the inclosed *gas*) were only 500 *lb.* supposing the barometer stood at 30 inches, and the specific gravity of mercury 13600?

Ans. 3805 *feet.*

132. If a conical frustum, the diameter of whose base is 2 feet, and height $2\frac{1}{4}$ feet, move in a fluid in the direction of its axis with the least end foremost, to find the diameter of that end when the resistance is the least possible.

Ans. 6 *inches.*

THE END.

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Errata in Vol. II.

Page 10, *l.* 4, for *ba* r. *bx*.

79, examp. 3, for $2^{\frac{1}{3}}$ r. $2^{\frac{1}{2}}$.

91, —— 14, for x, y, and x r. x, y, and z.

110, —— 1, for 19 r. 15 the value of x.

124, *l.* 13, for -1 r. $+1$.

208, examp. 5, Simp. equat. for $\sqrt{\left(-\frac{a^2}{2}\right)}$ r. $\sqrt{(-2a^2)}$.

213, *l.* 3, for 2 r. $1\frac{13}{16}$.

217, examp. 1, for $(2100\frac{1}{4})^4$ r. $(2100\frac{1}{4})^{\frac{1}{2}}$.

Errata in Vol. I.

Page	Line	*for*	*read.*
30	18	number	numbers
—	3 fr. bott.	in	is
34	11	$\frac{8}{14}$	$\frac{8}{11}$
38	7 fr. bott.	38526569	38526469
46	14	.012075	.012108 &c.
88	17	× 4	× 2
95	16	$11\frac{5}{7}$	$1\frac{5}{7}$
105	13	103 × 1.20	1.03 × 120
124	5	from	form
136	11	1300	1309 in some copies
141	7	1.588131	1.588134
160	read London for Oxford, and vice versâ, in the answer at bottom		
164	4 fr. bott.	one	once
165	11	21	23
168	14	$77\frac{54}{76}$	$76\frac{24}{76}$
—	15	$78\frac{8}{76}$	$76\frac{48}{76}$
169	20	$\frac{169}{365}$	$\frac{119}{365}$
204	5	rectangle	parallelogram
318	nat. sine of 43° 15′ is 0.6852; and BD = 274.7		
319	3 fr. bott.	478	278
357	2	RAB	RAG
364	18	72°	70°
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391	15	$36\frac{1}{2}$	$34\frac{1}{2}$
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418	7	.367	.376
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Errata in Vol. I.

Page	Line	*for*	*read.*
423	4 fr. bott.	42′ 14″	32′ 7″
		43′ 46″	53′ 53″.
427	5 fr. bott.	both	bore
431	9	$\frac{5}{9}$	$\frac{1}{6}$
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Log. of 6241 for 4254 *r.* 5254
of 6481 for 0642 *r.* 1642